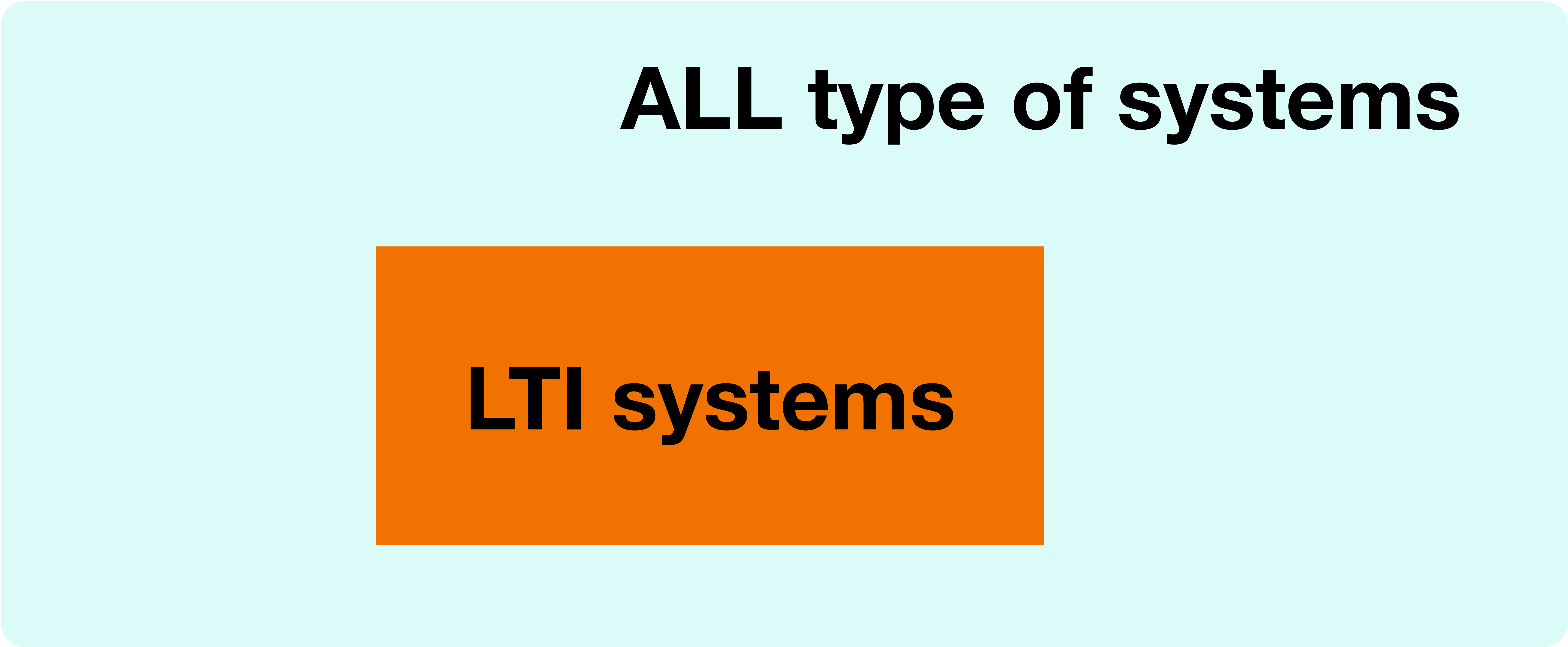
Topic 1- part 3 -
"LTI systems in time domain"

Discrete Time Systems (DTS)

Luca Martino — luca.martino@urjc.es — http://www.lucamartino.altervista.org

LTI systems

- We will focus on: LINEAR TIME INVARIANT (LTI) SYSTEMS
- LTI systems in time
- LTI systems in transformed domain (frequency domain etc.)



In this slides, WE WILL SEE:

HOW to express (IN the TIME DOMAIN) the output of:

1. LTI systems in (continuous) time

2. LTI systems in DISCRETE time

HOW to express (<u>IN the TIME DOMAIN</u>) the output of:

1. LTI systems <u>in (continuous) time</u>

(this is just a *gentle and quick recall* since you should know these concepts from another previous course…)

HOW to express the output of LTI systems <u>in TIME</u>

Continuous Time - 2 WAYS:

- **CONVOLUTION INTEGRAL:**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

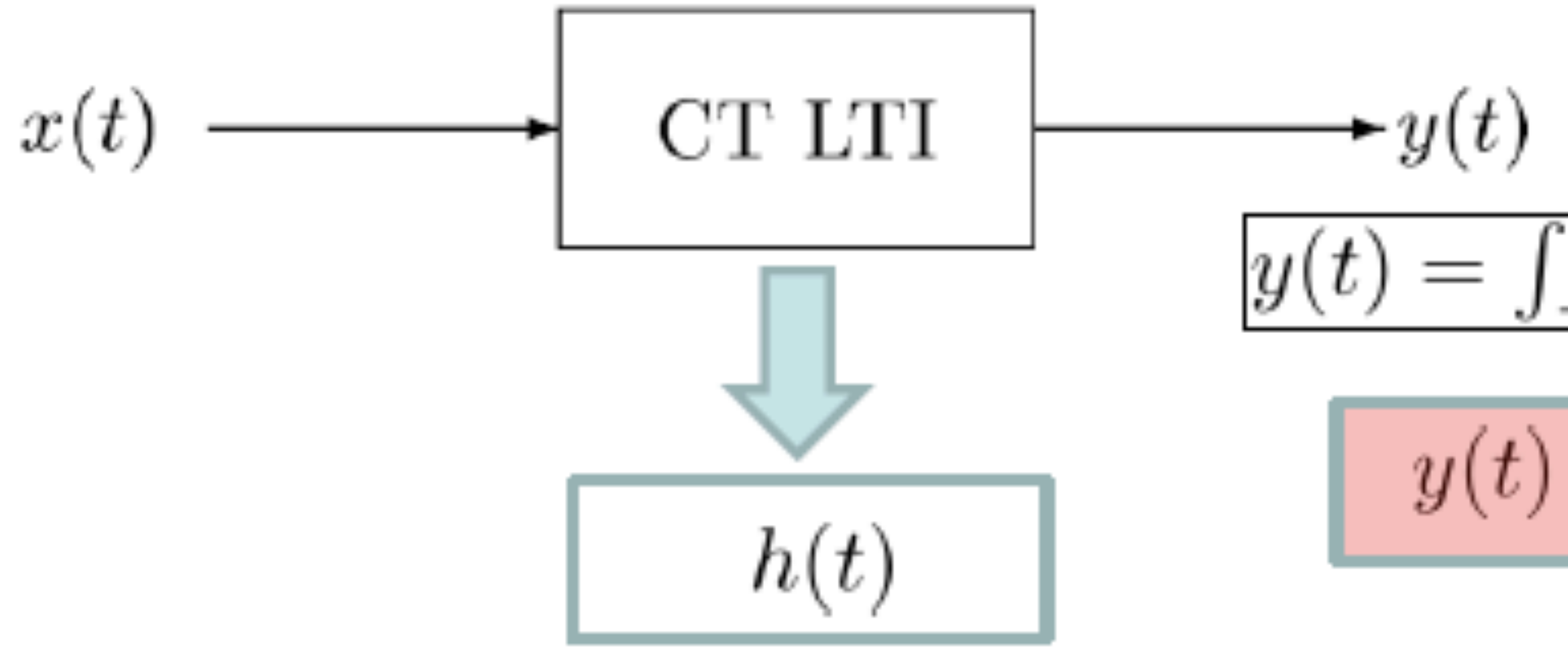
- **LINEAR DIFFERENTIAL EQUATIONS, WITH CONSTANT COEFFICIENTS AND NULL INITIAL CONDITIONS:**

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

(N) Initial conditions:

$$y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0^-} = \ldots = \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=0^-} = 0$$

LTI system in CT: convolution by integral



$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

$$\boxed{y(t) = x(t) * h(t)}$$

h(*t*)= impulse response (i.e., to the Dirac delta)

$$x(t) = \delta(t) \Longrightarrow y(t) = \int_{-\infty}^{+\infty} \delta(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} \delta(t-\tau)h(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\delta(t-\tau)d\tau = h(t)$$

h(*t*) represents completely the LTI system

Computation of the Convolution

- Calculating a convolution:
  - Convolution with a delta $\delta(t)$ → easy (we obtain the signal $x(t)$)
  - Convolution with exponentials → easy more or less (solution: another exp.)
  - Generic Convolution → more difficult

4 steps:

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$h(\tau) \xrightarrow[\text{Invert}]{} h(-\tau) \xrightarrow[\text{move}]{} h(t-\tau)$$

$$\xrightarrow[\text{Multiply}]{} x(\tau)h(t-\tau) \xrightarrow[\text{Integrate}]{} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Properties and examples of convolution

- Neutral Element is the unit impulse (Dirac Delta): $\delta(t)$

$$x(t) * \delta(t) = x(t) \quad \Longrightarrow \quad x(t) * \delta(t-t_0) = x(t-t_0)$$

- Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \quad \Longleftrightarrow \quad h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

- Distributive:

$$x(t) \longrightarrow \boxed{h_1(t) + h_2(t)} \longrightarrow y(t) = x(t) * \underbrace{[h_1(t) + h_2(t)]}_{h_{eq}(t)}$$

$$\Updownarrow$$

$$\begin{tikzcd}
 & \boxed{h_1(t)} \arrow[dr] & & \\
x(t) \arrow[ur] \arrow[dr] & & \oplus \arrow[r] & y(t) = x(t) * h_1(t) + x(t) * h_2(t) \\
 & \boxed{h_2(t)} \arrow[ur] & &
\end{tikzcd}$$

Properties and examples of convolution

- Associative:

$$y(t) = [x(t) * h_1(t)] * h_2(t) \Longleftrightarrow y(t) = x(t) * [h_1(t) * h_2(t)]$$

$$y(t) = x(t) * [h_2(t) * h_1(t)] \Longleftrightarrow y(t) = [x(t) * h_2(t)] * h_1(t)$$

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t) \Longleftrightarrow x(t) \rightarrow \underbrace{\boxed{h_1(t) * h_2(t)}}_{h_{eq}(t)} \rightarrow y(t)$$

Properties of LTI systems = become properties about *h*(*t*)

- Causality:

$$h(t) = 0, \quad \forall t < 0$$

- Stability:

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \rightarrow h(t)$$

- Without memory:

$$h(t) = 0, \quad \forall t \neq 0 \rightarrow y(t) = cte.x(t)$$

HOW to express (IN the TIME DOMAIN) the output of:

2. LTI systems in DISCRETE time

HOW to express the output of LTI systems in TIME

DISCRETE Time - THIS IS A SPOILER !!

- **CONVOLUTION SUM:**

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad y[n] = x[n] * h[n]$$

- **LINEAR *DIFFERENCE* EQUATIONS, WITH CONSTANT COEFFICIENTS AND NULL INITIAL CONDITIONS:**

$$\sum_{i=0}^{L} b_i y[n-i] = \sum_{r=0}^{R} c_r . x[n-r]$$

$n \geq 0$

$n = 0, 1, 2, 3...$

With L-INITIAL CONDITIONS (they are required)

$$y[-1], y[-2], ..., y[-L]$$

Linear Difference Equations

- A LTI systems in DT can be expressed using linear difference equations with constant coefficients. **(and null initial conditions)**

- Definition: $y[n] = \sum_{p=1}^{P} a_p y[n-p] + \sum_{m=0}^{M} b_m x[n-m]$

- They are ARMA (autoregressive-moving average) filters
- If all a_p=0 ➔ FIR (FINITE IMPULSE RESPONSE) filters
- If all b_m=0 except b_0 ➔ IIR (INFINITE IMPULSE RESPONSE) filters

Definition of LTI systems in DT

- We focus on linear and invariant systems in DT:

$$x[n] \rightarrow \boxed{\text{LTI systems}} \rightarrow y[n]$$

$$\text{If: } x[n] \rightarrow y[n]$$
$$\text{Then: } x[n-n_0] \rightarrow y[n-n_0]$$

$$\text{If: } x_1[n] \rightarrow y_1[n] \qquad x_2[n] \rightarrow y_2[n]$$
$$\text{Then: } ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

Response to the impulse in DT

- Impulso response:

$$\delta[n] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n]$$

For this reason, we call h[n] "impulse response".

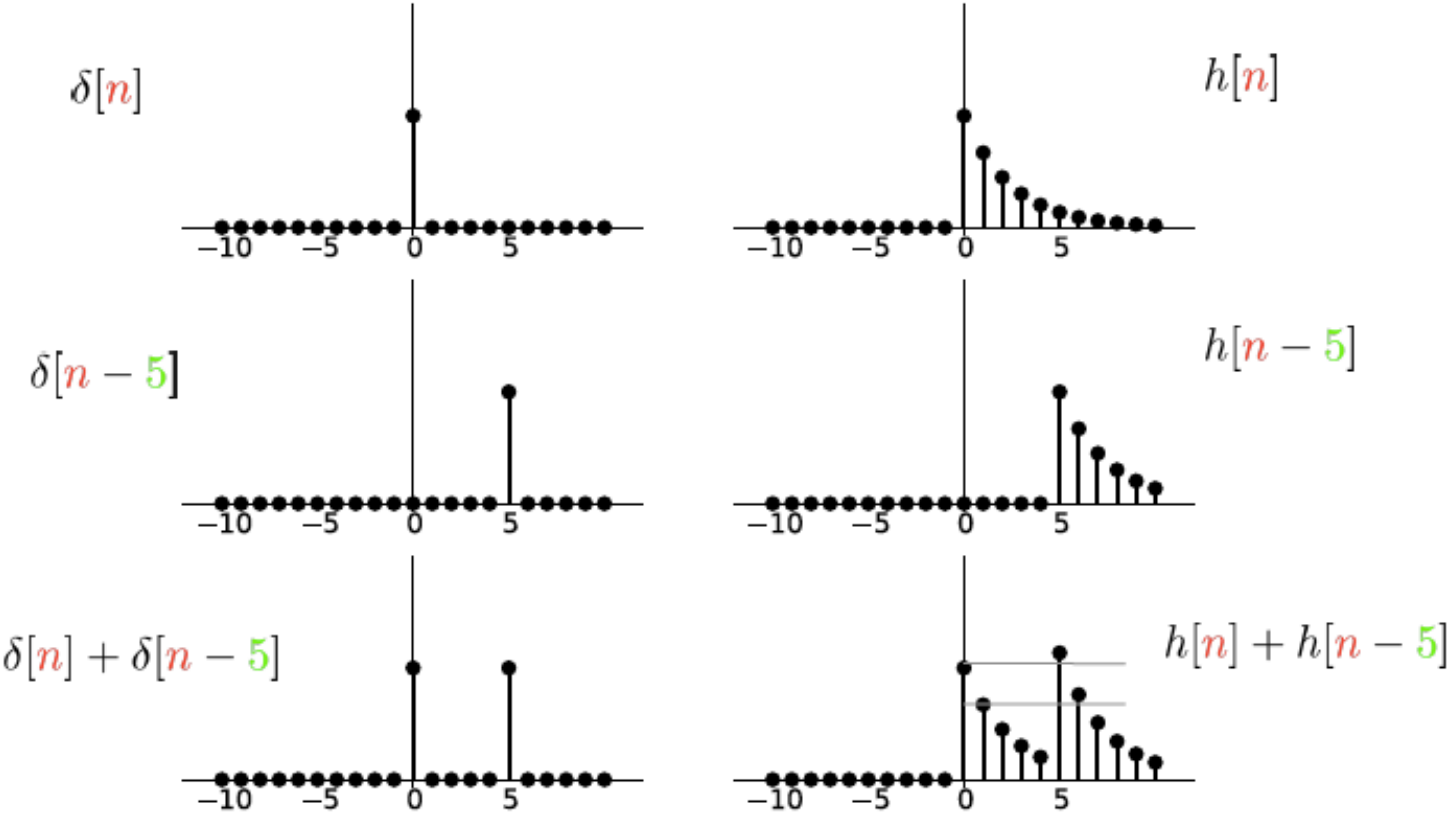
- For the time invariance:

$$\delta[n-n_0] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n-n_0]$$

- For the linearity:

$$\delta[n-n_0]+\delta[n-n_1] \longrightarrow \boxed{\text{SLTI}} \longrightarrow h[n-n_0]+h[n-n_1]$$

Response to the impulse in DT



Response to the impulse in DT

- Response to train of deltas ➔ sum of response to the impulse (for the linearity)

$$\sum_{k=-\infty}^{\infty} a_k \delta[n-k] \rightarrow \boxed{\text{SLTI}} \rightarrow \sum_{k=-\infty}^{\infty} a_k h[n-k]$$

- Since x[n] can be expressed as a train of deltas, then:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{\text{SLTI}} \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution in DT

- Hence the output y[n] can be obtained as the convolution of h[n] with x[n]:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad\qquad y[n] = x[n] * h[n]$$

- Convolution of two signals in DT; notation:

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Computing of the convolution

- For any time instant n: **(practical consideration)**

1) Express in the domain k: $h[k]$ $\qquad$ $x[k]$

2) Invert: $h[-k]$

3) Move n units: $h[n-k]$

4) Multiply: $x[k]h[n-k]$

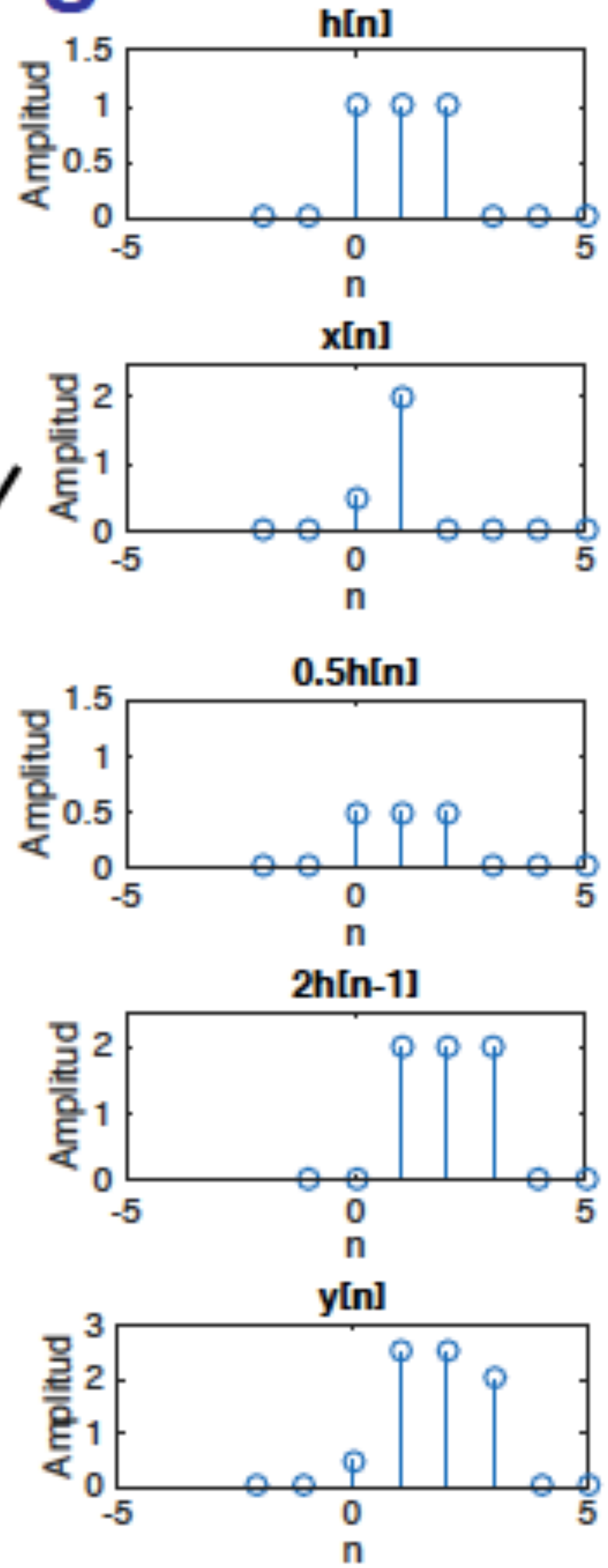
5) Sum:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Alternative way of computing a convolution

Alternative way:

1) We can express one signal as sum of deltas
2) Then we make the convolution with the other signal above in step 1)
3) We sum all the signals obtained in step 2)



In the example on the right ➔

We can express x[n] as:

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

$$y[n] = h[n] * x[n] = 0.5h[n] + 2h[n-1]$$

Length of the convolution is 4:

Start: sum of the starts (0+0=0)
End: sum of the ends (2+1=3)
(considering non-zero samples)

Length of the convolution is N+M-1

Properties of LTI systems in DT

- The response to the impulse h[n] provides a complete characterization of the LTI system.
- See below:
  - Causality: $h[n] = 0, \forall n < 0$
  - Stability: $\sum_{-\infty}^{\infty} |h[n]| < \infty$
  - Memory: $h[n] = 0, \forall n \neq 0$

Some additional properties of the LTI systems can be expressed as conditions over h[n]

Properties of the LTI systems in DT

- Distributive property, parallel systems:

$$y[n] = x[n] * \underbrace{[h_1[n] + h_2[n]]}_{h_{eq}(t)} = x[n] * h_1[n] + x[n] * h_2[n]$$

- Associative property, systems in series:

$$y[n] = x[n] * h_1[n] * h_2[n] = x[n] * \underbrace{h_2[n] * h_1[n]}_{h_{eq}(t)}$$

Questions?