Solved Problems - Zeta Transform
part 3

Linear systems and circuit applications
Discrete Time Systems

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Example 14

Consider the following signal:

$$x[n] = \cos(n)$$

(a) Compute the Zeta Transform: $X(z) = ?$

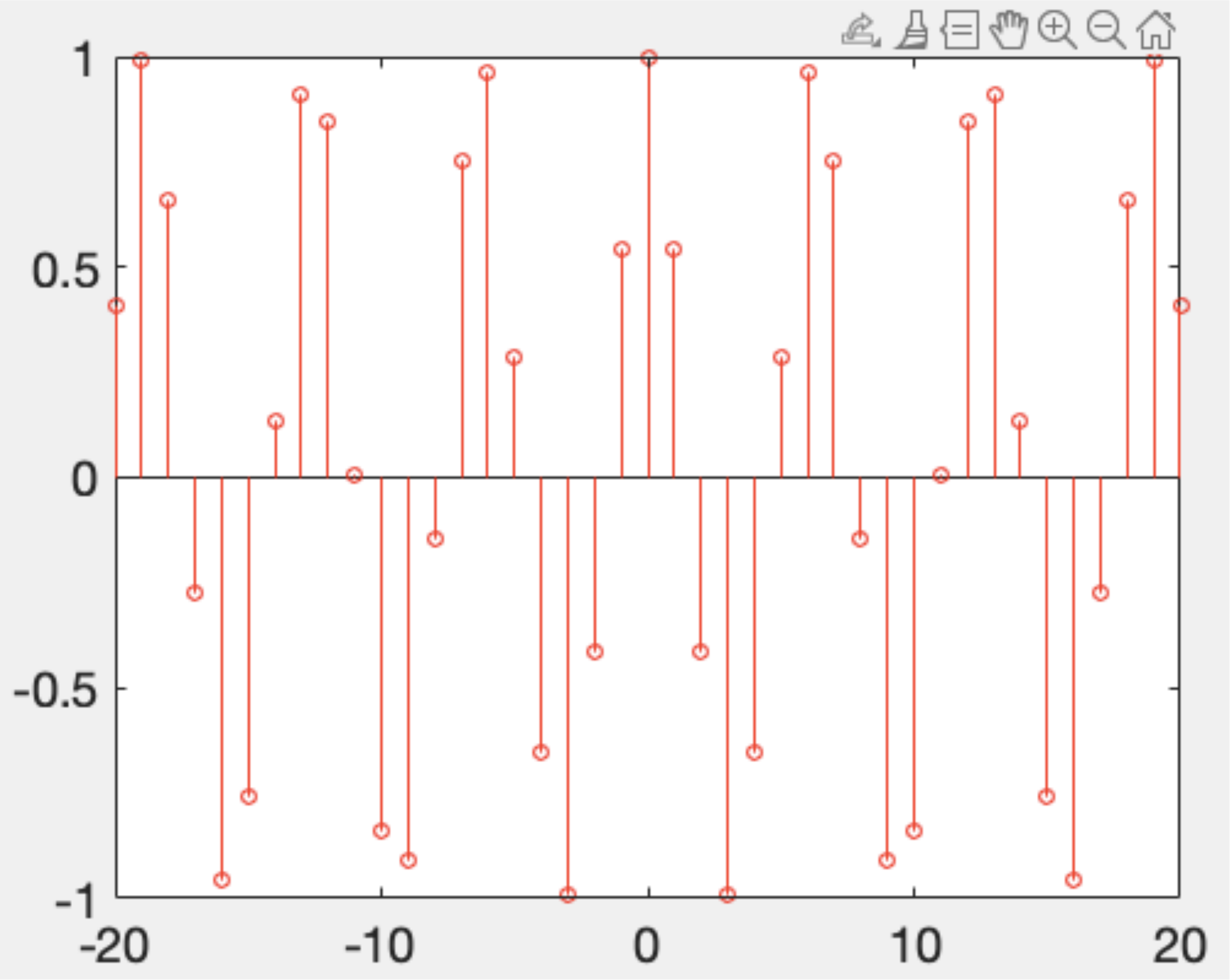
(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 14

(a) Thinking to the signal x[n]: the signal is growing as n grows, oscillating… but it is growing (even fast).



Example 14

This signal has a continuous/constant oscillation and has infinite energy.

Then, the Zeta Transform does not exists !!!

Example 14

The Zeta Transform does not exists !!!

(b) The Zeta Transform does not exists, then there are not “zeros”

(c) The Zeta Transform does not exists, then is like that all the complex planes is formed by “poles” (all points in the complex plan are poles…) (no voy a requirer está respuesta…diciendo que no existe ya se entiende el resto….)

(d) Therefore, the stand. FT does not exists neither since the Zeta transform is an extension of the stand. FT (which admits more signals…)

Example 15

Consider the following signal:

$$x[n] = 2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 15

(a) Using the direct definition:

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \\
&= \sum_{n=-\infty}^{0} 2^n z^{-n} + \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} \\
&= \sum_{k=0}^{\infty} 2^{-k} z^{k} + \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2} z\right)^k + \sum_{n=1}^{\infty} \left(\frac{1}{4} z^{-1}\right)^n \\
&= \frac{1}{1-\frac{1}{2}z} + \frac{1}{4} z^{-1} \frac{1}{1-\frac{1}{4}z^{-1}} \longrightarrow \text{ONLY IF } \left|\frac{1}{2}z\right| < 1 \text{ and } \left|\frac{1}{4}z^{-1}\right| < 1 \text{ jointly !!}
\end{aligned}
$$

Example 15

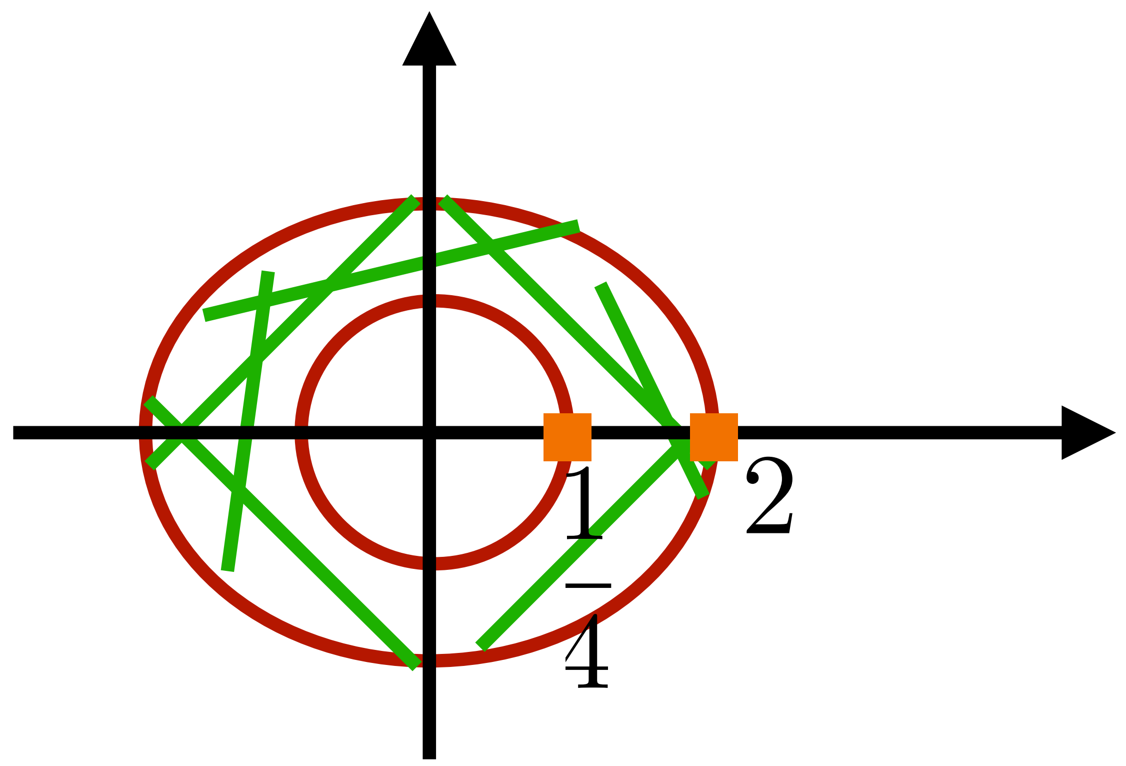
$$X(z) = \frac{1}{1 - \frac{1}{2}z} + \frac{1}{4}z^{-1}\frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$= \frac{1}{1 - \frac{1}{2}z} + \frac{1/4}{z - \frac{1}{4}}$$

$$= \frac{\frac{7}{8}z}{\left(1 - \frac{1}{2}z\right)\left(z - \frac{1}{4}\right)}$$

ONLY IF $\left|\frac{1}{2}z\right| < 1$ and $\left|\frac{1}{4}z^{-1}\right| < 1$ jointly !!

ROC: $|z| < 2$ and $|z| > \frac{1}{4}$ jointly !!



Example 15

summary:

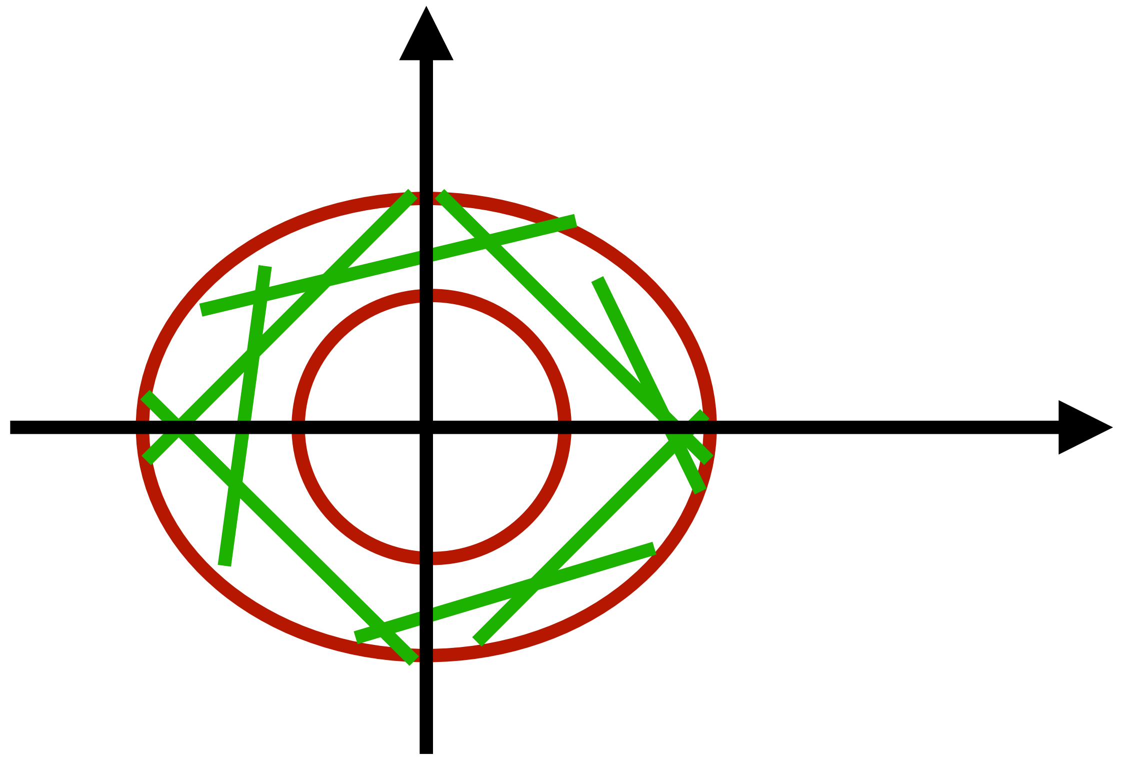
$$X(z) = \frac{\frac{7}{8}z}{\left(1 - \frac{1}{2}z\right)\left(z - \frac{1}{4}\right)}$$

$$\text{ROC: } |z| < 2 \text{ and } |z| > \frac{1}{4} \text{ jointly !!}$$

(b) zeros: two zeros, one at z=0 and one at z=Infinity.

(c) poles and ROC: two poles at z=2 and z=1/4,

$$\text{ROC: } |z| < 2 \text{ and } |z| > \frac{1}{4} \text{ jointly !!}$$



Example 15

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{\frac{7}{8}z}{\left(1-\frac{1}{2}z\right)\left(z-\frac{1}{4}\right)} \Longleftrightarrow X(\Omega) = \frac{\frac{7}{8}e^{j\Omega}}{\left(1-\frac{1}{2}e^{j\Omega}\right)\left(e^{j\Omega}-\frac{1}{4}\right)}$$

Example 16

Consider the following signal:

$$x[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Compute the Zeta Transform: $X(z) = ?$

Example 16

We can observe that the signal can be re-written as:

$$x[n] = \sum_{k=0}^{\infty} \delta[n-k] \longrightarrow x[n] = u[n]$$

and we have already computed (in other example) that

$$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\text{ROC: } \forall z \in \mathbb{C} : \text{ such that } |z| > 1$$

Example 17

Consider the following signal:

$$x[n] = \sum_{k=0}^{\infty} (-1)^k \delta[n-k]$$

Compute the Zeta Transform: $X(z) = ?$

Example 17

We can observe that the signal can be re-written as:

$$x[n] = \sum_{k=0}^{\infty} (-1)^k \delta[n-k] \longrightarrow x[n] = (-1)^n u[n]$$

$$x[n] = (-1)^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} (-1)^n \cdot z^{-n} = \sum_{n=0}^{\infty} \underbrace{(-z^{-1})^n}_{|-z^{-1}|<1 \;\Downarrow\; |z|>1} = \frac{1}{1+z^{-1}}, \quad |z|>1$$

Example 18

Diseñar sistema discreto, de forma que: en instante n, salida = suma de entrada en n-1, n-3 y n-5.

a) $H(z)$? Representar diagrama de polos y ceros, ROC.

b) $h[n]$? Estable, causal?

c) Cumple requisitos un SLTI y causal **?**

(d) Compare with the system represented by $H(z) = \dfrac{z^6 - 1}{z^7 - z^5}$

Are they the same system? explain.

Example 18

a) $y[n] = x[n-1] + x[n-3] + x[n-5]$

T.Z. ↳

$$Y(z) = X(z)\left[z^{-1} + z^{-3} + z^{-5}\right]$$

$$\boxed{H(z)} = \frac{Y(z)}{X(z)} = \left[z^{-1} + z^{-3} + z^{-5}\right] = \frac{1}{z^{1}} + \frac{1}{z^{3}} + \frac{1}{z^{5}} = \frac{z^4 + z^2 + 1}{z^5}$$

Polo: $z=0$ (de orden 5)

Example 18

z=Infinity is one zero, and the rest of 4 zeros are the solutions of the equation:

$$\text{Zeros: } z^4 + z^2 + 1 = 0$$

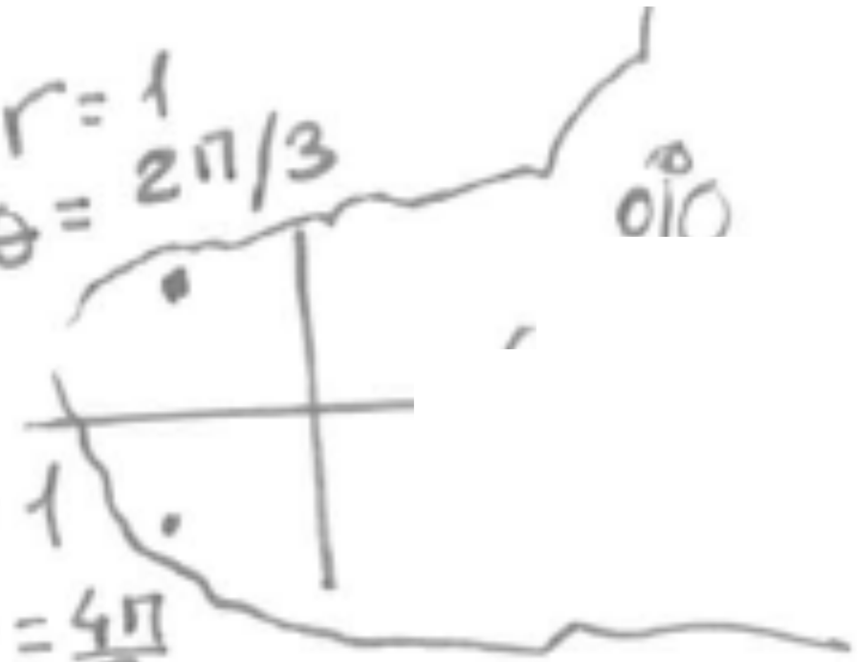
$$\text{setting } t = z^2 \rightarrow t^2 + t + 1 = 0$$

$z = 0$ (de orden 5)

$t = z^2 \rightarrow t^2 + t + 1 = 0 \rightarrow t = - \frac{\pm\sqrt{(-1)^2 - 4}}{2} =$

$\nearrow -\frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \begin{cases} r = 1 \\ \theta = 2\pi/3 \end{cases}$

$\searrow -\frac{1}{2} - j\frac{\sqrt{3}}{2} \quad \begin{cases} r = 1 \\ \theta = \frac{4\pi}{3} \end{cases}$



Example 18

$$t_1 = e^{j\frac{2\pi}{3}}$$

$$t_2 = e^{j\frac{4\pi}{3}}$$

In this specific case, you could also do

$$z = \sqrt{t} \quad z = -\sqrt{t}$$

obtaining also the symmetric solution (with respect to the zero-origin point; i.e., odd-symmetry)

Solving $t = z^2$ in the complex plane:

Recall that $e^{j2\pi k} = 1$

$$z = \left(e^{j\frac{2\pi}{3}+2\pi k}\right)^{1/2}$$

$$z = \left(e^{j\frac{4\pi}{3}+2\pi k}\right)^{1/2}$$

$$\rightarrow$$

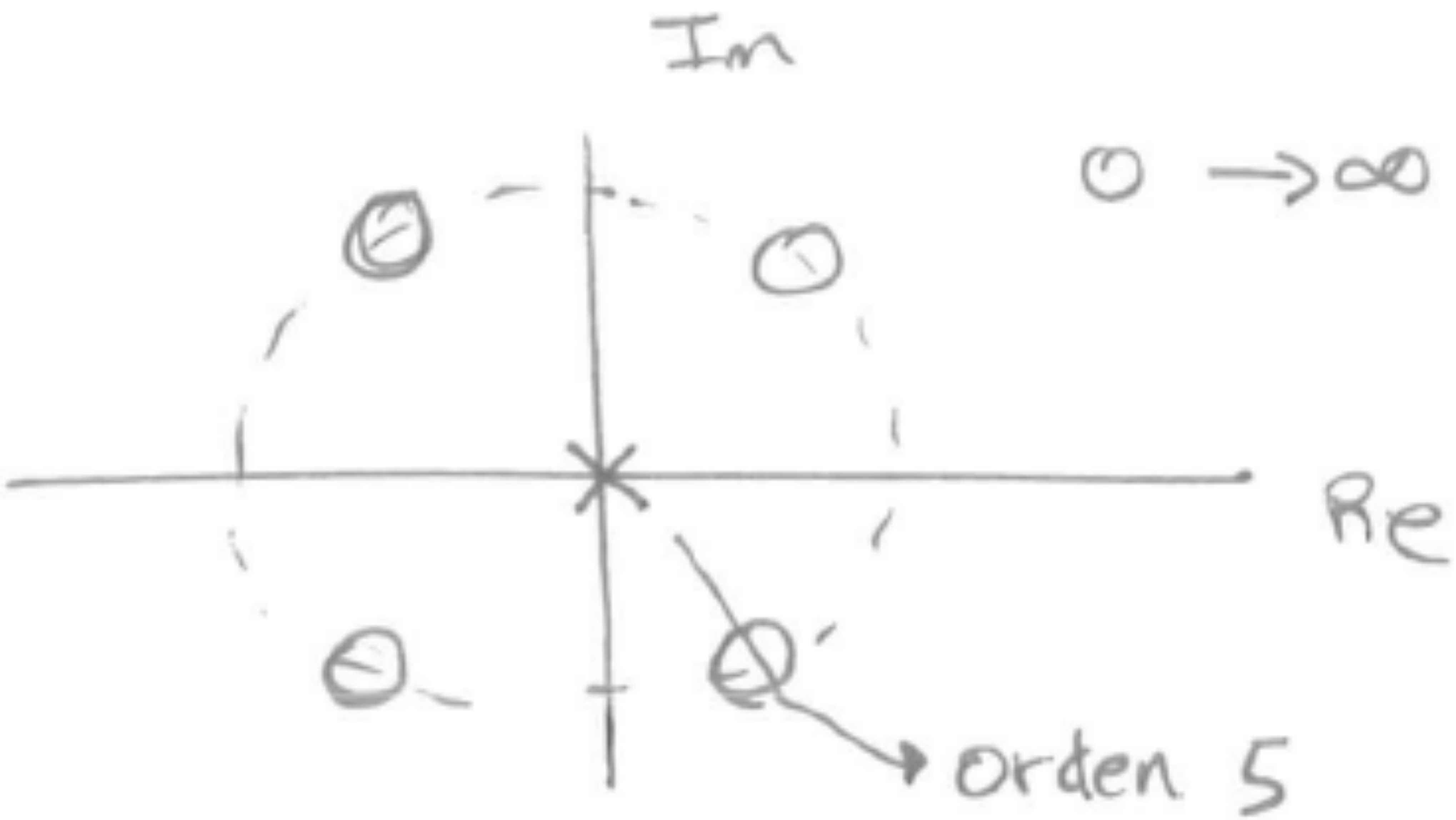
$$z = e^{j\frac{\pi}{3}+\pi k}$$

$$z = e^{j\frac{2\pi}{3}+\pi k}$$

$$k = 0, 1, 2, 3...$$

Example 18

Varying k=0,1,2,3,… the only different points are 4:



ROC : todo plano z, excepto origen (todos polos en z=0).

Example 18

b) $h[n] = \delta[n-1] + \delta[n-3] + \delta[n-5]$

Mirando $H(z)$ con propiedades (tablas)

CAUSAL : $h[n] = 0 \,,\quad n < 0$

ESTABLE : ROC incluye Círculo Unitario.

Example 18

c)

$$H(z) = \frac{z^6 - 1}{z^7 - z^5} = \frac{z^6 - 1}{z^5 \cdot (z^2 - 1)}$$

After a look we can directly observe that:

- two zeros z=-1 and z=1 (and another zero is z=\infinity)

- there are other 4 zeros (see next slide)

- two poles are z=-1 and z=1; another pole is z=0 (of order 5)

Example 18

- there are other 4 zeros; in fact, we can write:

$$z^6 - 1 = (z+1)(z-1)(z^4 + z^2 + 1)$$

$$z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$$

this can be obtained:

$$
\begin{array}{r|l}
z^6 - 1 & z^2 - 1 \\ \cline{2-2}
-z^6 + z^4 & z^4 + z^2 + 1 \\ \cline{1-1}
z^4 - 1 & \\
-z^4 + z^2 & \\ \cline{1-1}
z^2 + 1 & \\
-z^2 - 1 & \\ \cline{1-1}
0 &
\end{array}
$$

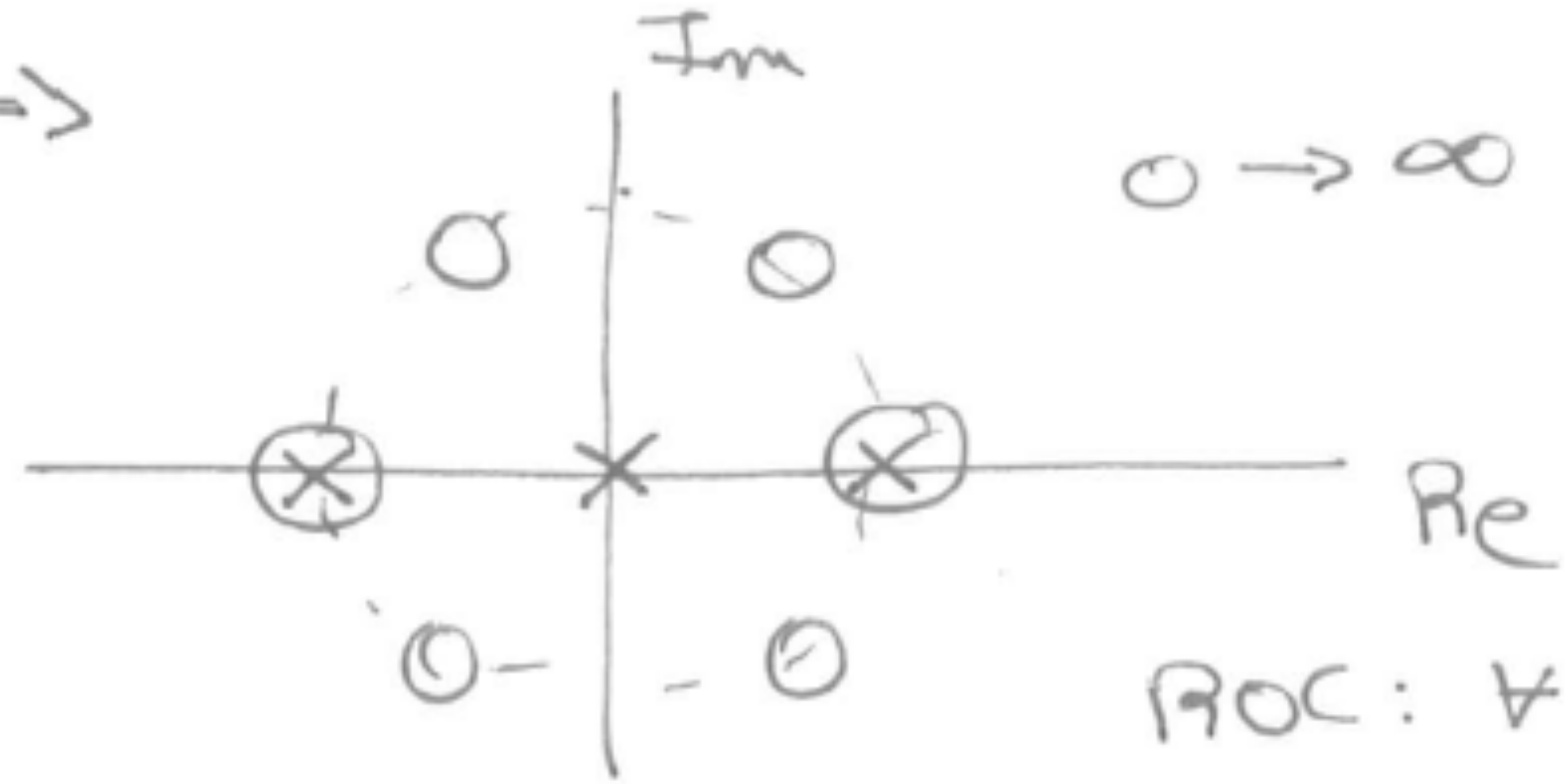
Example 18

- there are other 4 zeros, solution AGAIN of this equation:

$$\text{Zeros: } z^4 + z^2 + 1 = 0$$

• Los polos son cancelados por los ceros

$\Rightarrow$



ROC: $\forall\ z$ excepto origen.

$\Rightarrow$ Como ambos sistemas tienen igual diagrama polos y ceros y misma ROC $\Rightarrow$ Sí cumple requisitos de enunciado.

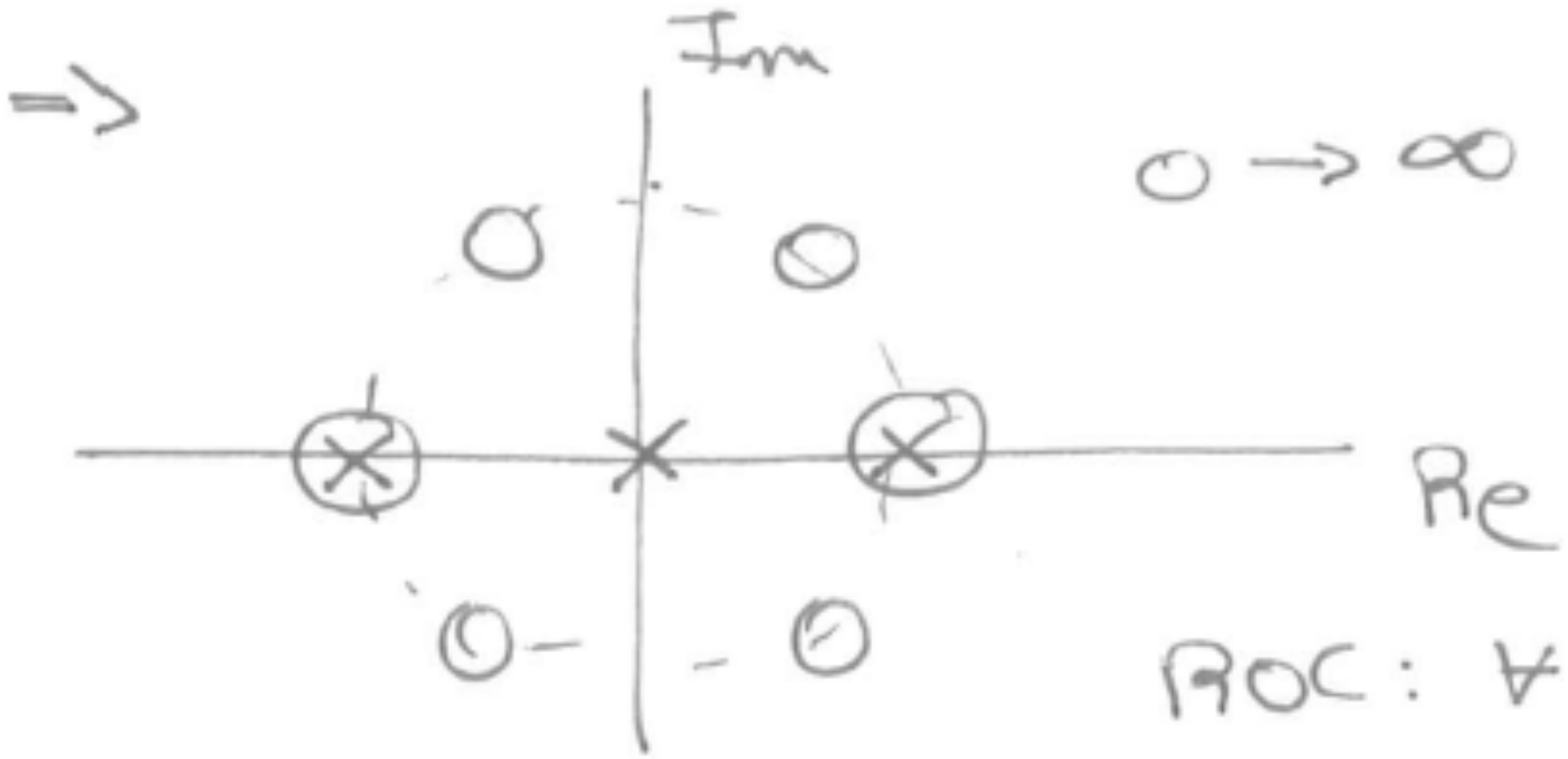
Example 18

- there are other 4 zeros, solution AGAIN of this equation:

$$\text{Zeros: } z^4 + z^2 + 1 = 0$$

• Los polos son cancelados por los ceros

⇒



ROC: ∀ z excepto origen.

⇒ Como ambos sistemas tienen igual diagrama polos y ceros y misma ROC ⇒ Sí cumple requisitos de enunciado.

Example 19

Consider Zeta transforms of the input x[n] and output y[n] of a system:

$$X(z) = 1 - z^2$$

$$Y(z) = z$$

find H(z) of the impulse response h[n].

Example 19

$$X(z) = 1 - z^2$$

$$Y(z) = z$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{1 - z^2}$$

Example 20

Consider Zeta transforms of the input x[n] and the impulse response h[n] of a system:

$$X(z) = 1 - z^2$$
$$H(z) = z - 3$$

find Y(z) of the output y[n] of the system.

Example 20

$$X(z) = 1 - z^2$$
$$H(z) = z - 3$$

$$Y(z) = H(z)X(z) = (z-3)(1-z^2)$$

Example 20

Consider the system with the following poles and zeros:

POLOS:

$z_{p_1} = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}$

$z_{p_2} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$

$z_{p_3} = 0'8$

CEROS:

$z_{c1} = e^{j\frac{\pi}{2}}$

$z_{c2} = e^{-j\frac{\pi}{2}}$

$z_{c3} = -1'2$

(a) say if the system is stable.

(b) write the difference equation corresponding to the system.

Example 21

Consider the causal system with the following poles and zeros:

POLOS:

$$z_{p_1} = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}$$

$$z_{p_2} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$$

$$z_{p_3} = 0'8$$

CEROS:

$$z_{c1} = e^{j\frac{\pi}{2}}$$

$$z_{c2} = e^{-j\frac{\pi}{2}}$$

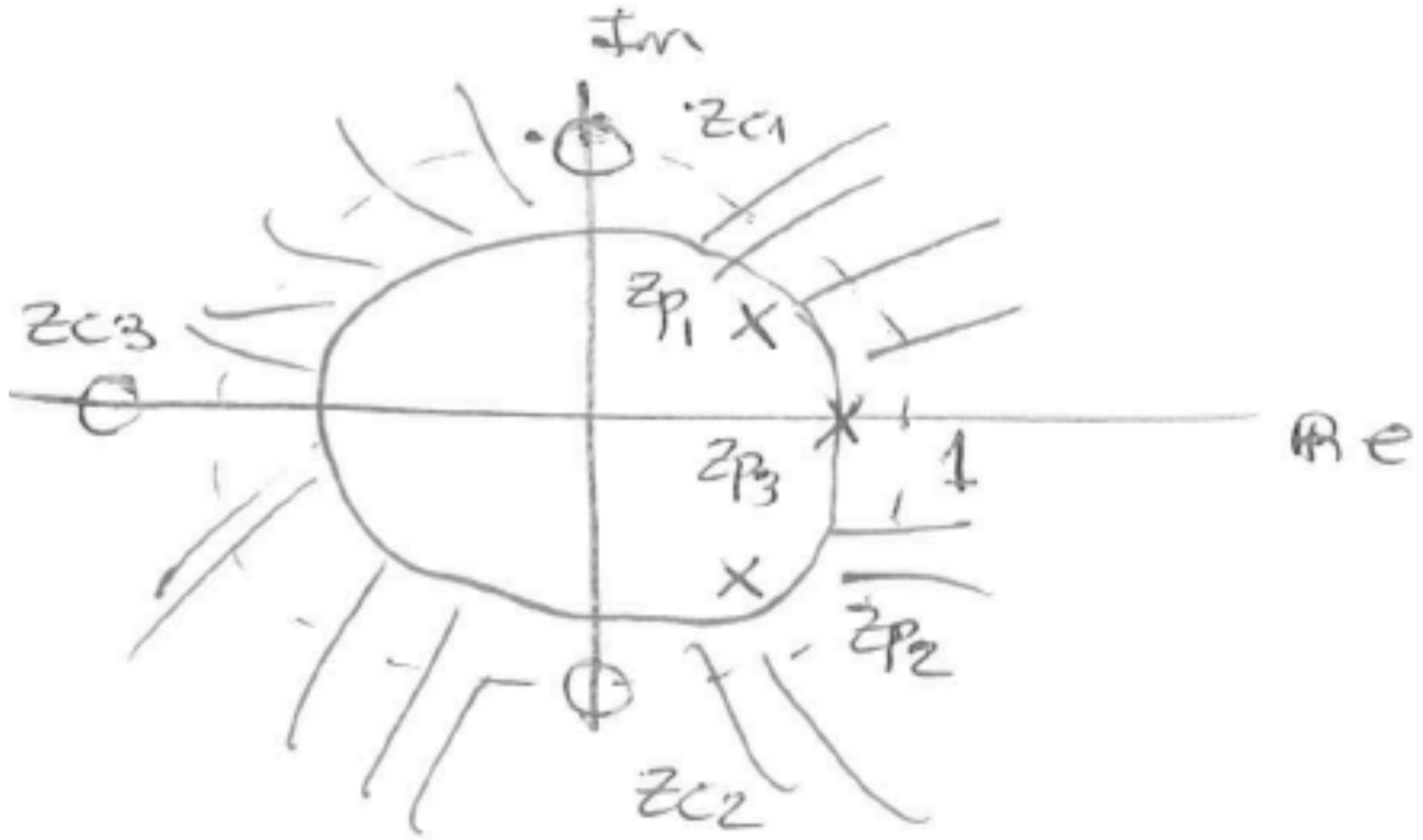
$$z_{c3} = -1'2$$

(a) say if the system is stable.

(b) write the difference equation corresponding to the system.

Example 21

x-marks ==> poles, circles ==> zeros



POLOS:

$z_{p_1} = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}$

$z_{p_2} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$

$z_{p_3} = 0'8$

CEROS:

$z_{c_1} = e^{j\frac{\pi}{2}}$

$z_{c_2} = e^{-j\frac{\pi}{2}}$

$z_{c_3} = -1'2$

a) Estable?

Como es causal, ROC = circunf. cuyo radio se corresponde al valor del módulo del polo más alejado del origen.

Como contiene C.U. ⇒ ESTABLE.

$\boxed{\text{ROC: } |z| > 0'8}$

Example 21

Ec. en diferencias ?

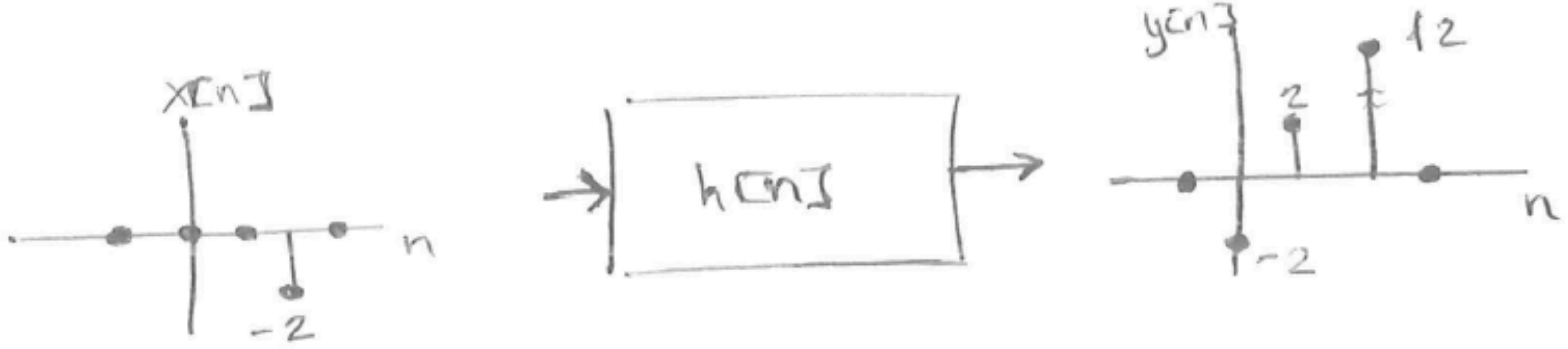
$$Y(z)\,(1-1'8z^{-1}+1'3z^{-2}-0'4z^{-3}) = X(z)\,(1+1'2z^{-1}+z^{-2}+1'2z^{-3})$$

$T.z^{-1}$

$$y[n]-1'8\,y[n-1]+1'3\,y[n-2]-0'4\,y[n-3] = x[n]+x[n-1]1'2+$$
$$+\,x[n-2]+1'2\,x[n-3]$$

Example 22

Un SLTI responde con $y[n]$ cuando se aplica $x[n]$:



a) Obtener, sin emplear transformadas, $h[n]$ y ec. en diferencias que caracterizan el sistema.

b) ¿Se trata de un sistema causal? Razonar respuesta.

c) Obtener $H(z)$. Dibujar diagrama polos y ceros y ROC.

Example 22

a) $x[n] = -2\,\delta[n-2]$

$y[n] = -2\,\delta[n] + 2\,\delta[n-1] + 12\,\delta[n-2]$

$y[n] = x[n] * h[n] = [-2\delta[n-2]] * h[n] =$

$= -2\delta[n] + 2\delta[n-1] + 12\,\delta[n-2]$

Example 22

then: $\Rightarrow h[n] = \delta[n+2] - \delta[n+1] - 6\cdot\delta[n]$

Cada término hace que desplazado a $(n-2)$, valor de $x[n]$, me de el resultado $y[n]$.

$x_0[n] = \delta[n] \rightarrow y_0[n] = h[n]$

$x_1[n] = \delta[n-2] \rightarrow y_1[n] = h[n-2]$

$x_2[n] = -2\cdot\delta[n-2] \rightarrow y_2[n] = -2h[n-2]$

Example 22

Ec. en diferencias: $\boxed{y[n] = x[n] * [\delta[n+2] - \delta[n+1] - 6\delta[n]] = x[n+2] - x[n+1] - 6x[n]}$

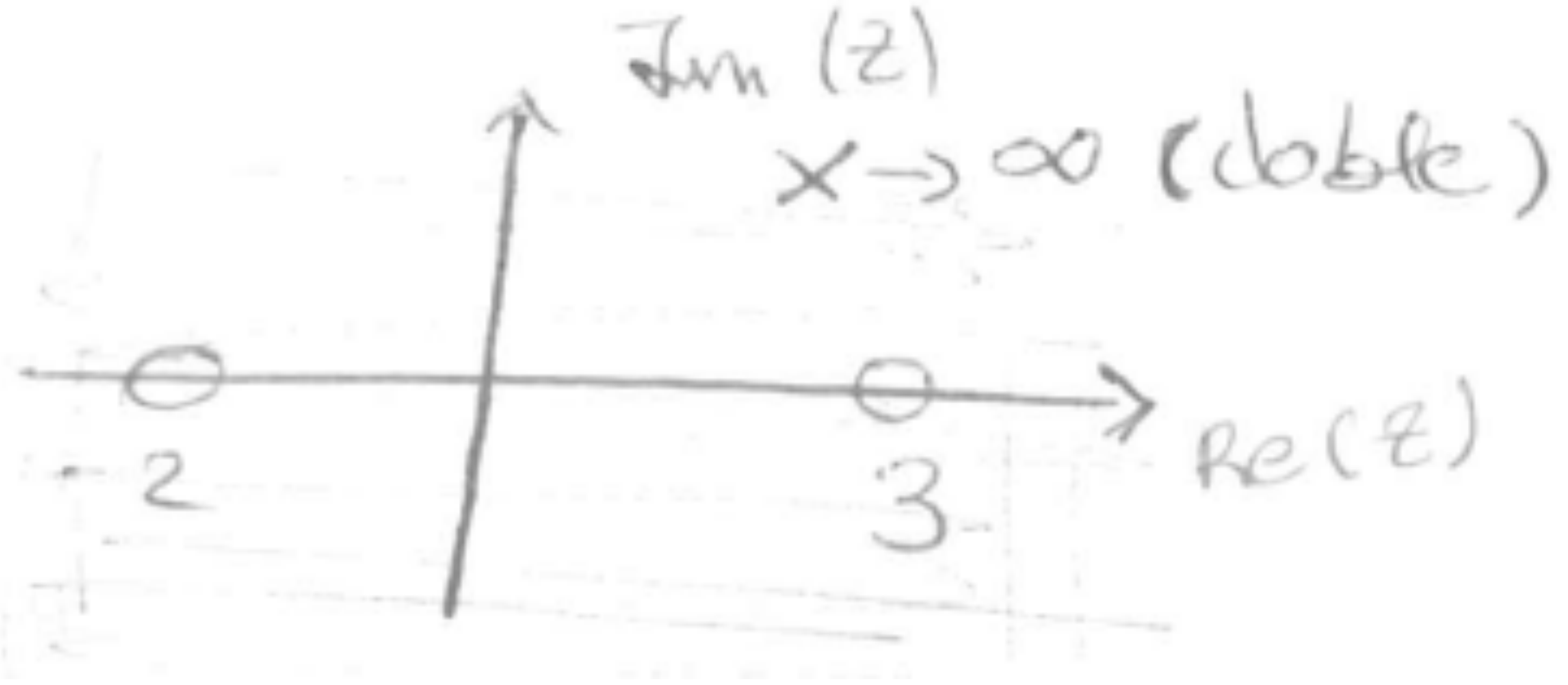
Example 22

b) El sistema es no causal → Salida en instante n, depende de la entrada en instantes posteriores.

Example 22

c) $H(z) = z^2 - z - 6$

$$\begin{cases} \text{Polos:} \quad z_{p_1} = z_{p_2} \to \infty \\ \text{Ceros:} \begin{cases} z_{c_1} = 3 \\ z_{c_2} = -2 \end{cases} \end{cases}$$



$$\boxed{\text{ROC} \quad \forall\, z \ \text{excepto} \ z \to \infty}$$

Example 23

Consider the following observations:

- x[n] is real and a right-sided signal

- X(z) has exactly 2 poles

- X(z) has a 2 zeros at z=0

- One pole of the two poles is at $z = \frac{1}{2}e^{j\frac{\pi}{2}}$

- X(1)=8/3, i.e., X(z)=8/3 at z=1

Find X(z).

Example 23

Since the signal is real and has two poles, the second pole must be the conjugate of

$$z = \frac{1}{2}e^{j\frac{\pi}{2}}$$

then, the other pole is :

$$z = \frac{1}{2}e^{-j\frac{\pi}{2}}$$

Thus, we have already all the poles. Infinity is not one of them.

Example 23

Since we also know that it has only 2 zeros at z=0, we can write:

$$
\begin{aligned}
X(z) &= A\frac{z^2}{(z-0.5e^{j\pi/2})(z-0.5e^{-j\pi/2})} \\
&= A\frac{z^2}{(z-0.5j)(z+0.5j)} \\
&= A\frac{z^2}{z^2+\frac{1}{4}}
\end{aligned}
$$

where A is a constant that we have to obtain.

Example 23

considering the last condition:

$$X(1) = \color{red}{A}\frac{1^2}{1^2 + \frac{1}{4}} = \frac{8}{3}$$

$$A = \frac{8}{3}\frac{5}{4} = \frac{10}{3}$$

Questions?