Solved Problems - Zeta Transform

Linear systems and circuit applications

Discrete Time Systems

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Example 1

Consider the following signal:

$$x[n] = \delta[n+5]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 1

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta[n+5] z^{-n} = z^{-(-5)} = z^{5}$$

$$X(z) = z^{5}$$

Example 1

(b) Zeros of X(z):

$$X(z) = z^5 = 0 \Longrightarrow z = 0$$

a zero of order 5 at z=0 (5 coincident zeros at z=0).

Example 1

(c) Poles of X(z):

$$X(z) = z^5$$

We have a multiple pole (of order 5) at infinity.

The the ROC is all the complex plane except infinity !!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C} \setminus \{\infty\}$$

Example 1

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = z^5 \Longrightarrow X(\Omega) = (e^{j\Omega})^5 = e^{j5\Omega}$$

Example 2

Consider the following signal:

$$x[n] = \delta[n-5]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 2

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta[n-5]z^{-n} = z^{-5}$$

$$X(z) = z^{-5} = \frac{1}{z^5}$$

Example 2

(b) Zeros of X(z):

$$X(z) = \frac{1}{z^5}$$

we have multiple zero (of order 5) at z=Infinity.

Example 2

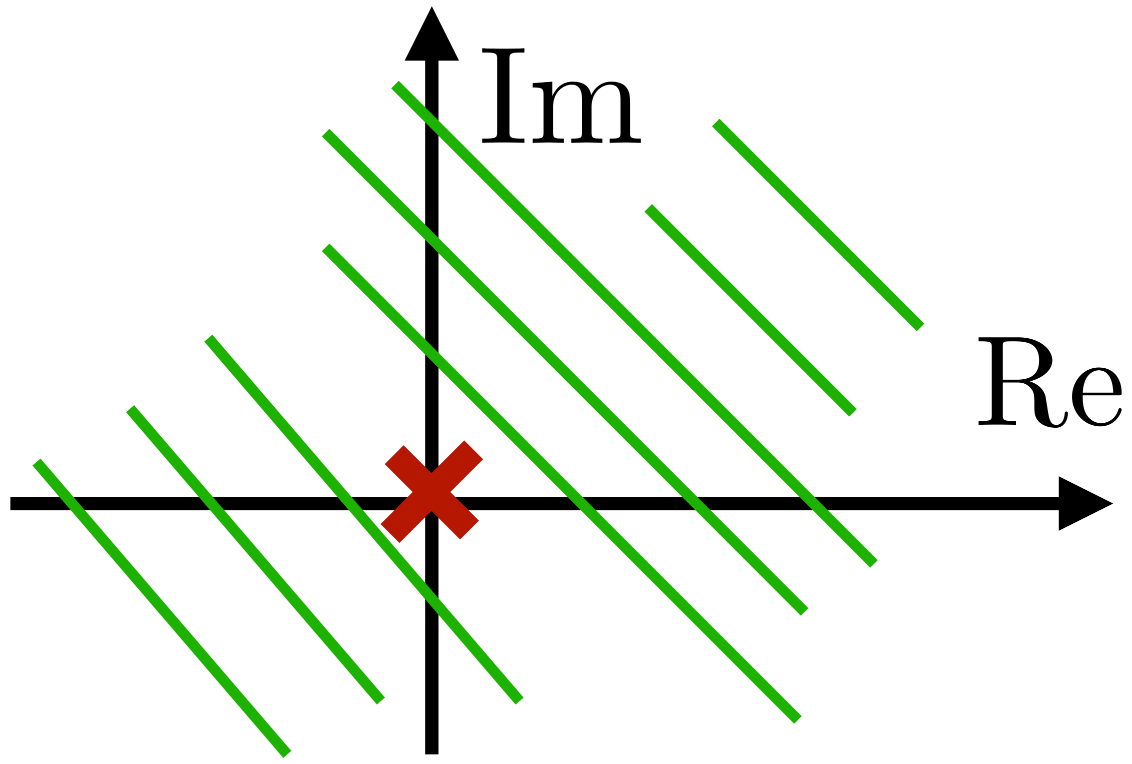
(c) Poles of X(z):

$$X(z) = \frac{1}{z^5}$$

We have a multiple pole (of order 5) at z=0.

The the ROC is all the complex plane except zero !!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C} \backslash \{0\}$$



Example 2

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = z^{-5} \Longrightarrow X(\Omega) = (e^{j\Omega})^{-5} = e^{-j5\Omega} = \frac{1}{e^{j5\Omega}}$$

Example 3

Consider the following signal:

$$x[n] = \delta[n]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 3

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta[n]z^{-n} = z^{-0} = 1$$

$$X(z) = 1$$

Example 3

(b) Zeros of X(z):

$$X(z) = 1$$

There are no zeros of X(z)!

Example 3

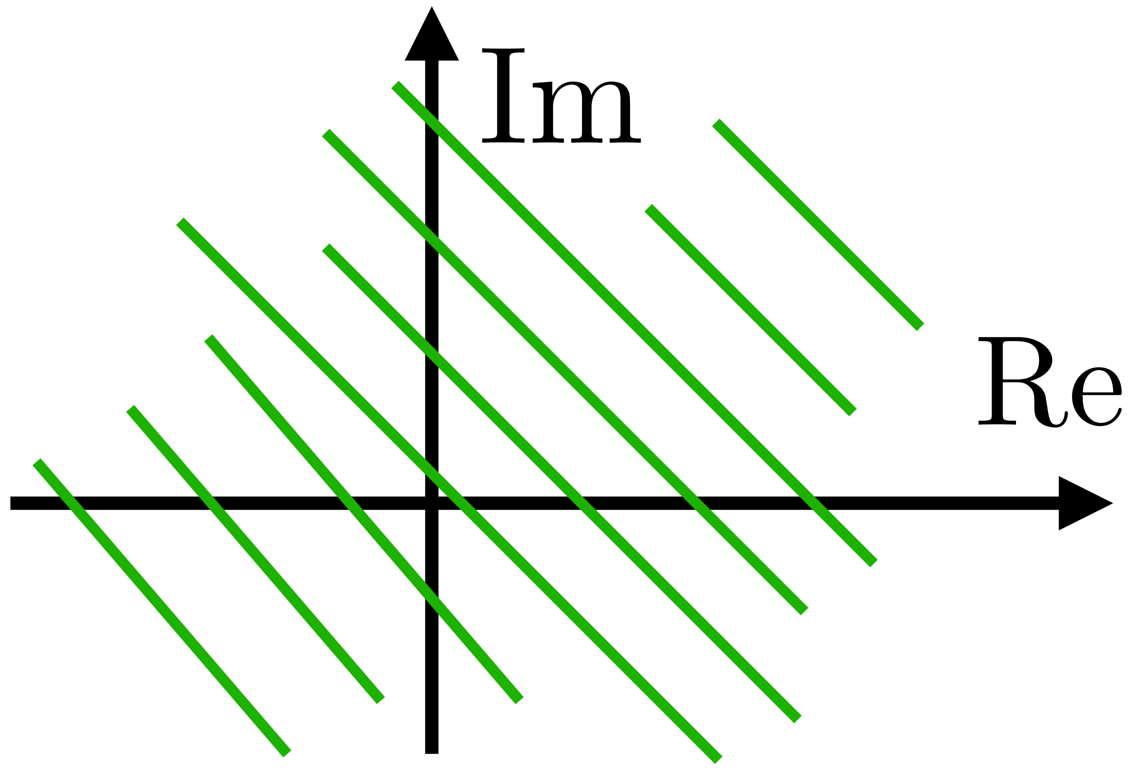
(c) Poles of X(z):

$$X(z) = 1$$

We have no poles!!!

The the ROC is all the complex plane!!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C}$$



Example 3

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = 1 \Longrightarrow X(\Omega) = 1$$

Example 4

Consider the following signal:

$$x[n] = \delta[n+5] + \delta[n-5]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 4

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} (\delta[n-5] + \delta[n+5])z^{-n} = z^{-5} + z^{5}$$

$$X(z) = \frac{1}{z^5} + z^5 = \frac{1+z^{10}}{z^5}$$

Example 4

(b) Zeros of X(z):

$$X(z) = \frac{1 + z^{10}}{z^5}$$

The zeros are:

- **10 different zeros (of order 1) which are the solutions in the complex plane of the equation:**

$$1 + z^{10} = 0 \Longrightarrow z^{10} = -1$$

Example 4

$$X(z) = \frac{1 + z^{10}}{z^5}$$

Why infinity is not a zero? since

$$\lim_{z \to +\infty} X(z) \approx \frac{z^{10}}{z^5} = z^5 = \infty$$

Example 4

$$z^{10} = -1$$

Recall that $e^{j2\pi k} = 1$

and $e^{j\pi(2k+1)} = e^{j2\pi k+\pi} = -1$

$$z^{10} = e^{j\pi(2k+1)}$$

$$z = \left(e^{j\pi(2k+1)}\right)^{1/10}$$

$$z = e^{j\frac{\pi(2k+1)}{10}} \qquad \text{for} \qquad k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

$$z = e^{j\frac{\pi}{10}}, z = e^{j\frac{3\pi}{10}}, z = e^{j\frac{\pi}{2}} = j, z = e^{j\frac{7\pi}{10}}, z = e^{j\frac{9\pi}{10}}, z = e^{j\frac{11\pi}{10}}, z = e^{j\frac{13\pi}{10}}, z = e^{j\frac{3\pi}{2}} = -j, z = e^{j\frac{17\pi}{10}}, z = e^{j\frac{19\pi}{10}}$$

Note that $e^{j\frac{21\pi}{10}} = e^{j\frac{\pi}{10}}$ and for all the z above $z^{10} = -1.$

Example 4

(c) Poles of X(z):

$$X(z) = \frac{1 + z^{10}}{z^5}$$

We have a multiple pole (of order 5) at z=0 (5 coincident poles at z=0) and a multiple pole (of order 10) at z=Infinity (5 coincident poles)

$$\lim_{z \to +\infty} X(z) \approx \frac{z^{10}}{z^5} = z^5 = \infty$$

The the ROC is all the complex plane except zero and Infinity!!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C} \backslash \{\infty\} \cup \{0\}$$

Example 4

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{1 + z^{10}}{z^5} \Longrightarrow X(\Omega) = \frac{1 + e^{j10\Omega}}{e^{j5\Omega}}$$

Example 5

Consider the following signal:

$$x[n] = \delta[n+5] + \delta[n] + \delta[n-5]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform (at least say how many).

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 5

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} (\delta[n-5] + \delta[n] + \delta[n+5])z^{-n} = z^5 + 1 + z^{-5}$$

$$X(z) = 1 + \frac{1}{z^5} + z^5 = \frac{z^5 + 1 + z^{10}}{z^5}$$

$$X(z) = \frac{z^{10} + z^5 + 1}{z^5}$$

Example 5

(b) Zeros of X(z):

$$X(z) = \frac{z^{10} + z^5 + 1}{z^5}$$

The zeros are:

- **10 different zeros (of order 1) which are the solutions in the complex plane of the equation:**

$$1 + z^5 + z^{10} = 0$$

Note that again:

$$\lim_{z \to +\infty} X(z) \approx \frac{z^{10}}{z^5} = z^5 = \infty$$

Example 5

(c) Poles of X(z):

$$X(z) = \frac{z^{10} + z^5 + 1}{z^5}$$

We have a multiple pole (of order 5) at z=0 (5 coincident poles at z=Infinity) and a multiple pole (of order 5) at z=Infinity (5 coincident poles at z=Infinity).

The the ROC is all the complex plane except at zero and at Infinity!!! Namely, in formula:

$$\text{ROC: } \forall z \in \mathbb{C} \backslash \{\infty\} \cup \{0\}$$

Example 5

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has finite energy, hence it admits standard FT. Then, we can compute the stand. FT by the definition or by the property of a delta (its FT is a complex exponential in frequency).

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? YES, then standard FT exists. Moreover, we know that setting r=1 in the Zeta Transform, we obtain the FT, i.e.,

$$z = re^{j\Omega} \Longrightarrow r = 1 \Longrightarrow z = e^{j\Omega}$$

$$X(z) = \frac{1 + z^5 + z^{10}}{z^5} \Longrightarrow X(\Omega) = \frac{1 + e^{j5\Omega} + e^{j10\Omega}}{e^{j5\Omega}}$$

Example 6

Consider the following signal:

$$x[n] = 1$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 6

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} z^{-n}$$

$$= \infty$$

(comparar con la serie geometrica...)

The Zeta Transform does not exists !!!

Example 6

The Zeta Transform does not exists !!!

(b) The Zeta Transform does not exists, then there are not “zeros”

(c) The Zeta Transform does not exists, then is like that all the complex planes is formed by “poles” (all points in the complex plan are poles…) (no voy a requirer está respuesta…diciendo que no existe ya se entiende el resto….)

(d) Therefore, the stand. FT does not exists neither since the Zeta transform is an extension of the stand. FT (which admits more signals…) - This signal admits Fourier Series (as a degenerate periodic signal - with infinite period) and Generalized FT.

Example 7

Consider the following signal:

$$x[n] = u[n]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 7

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

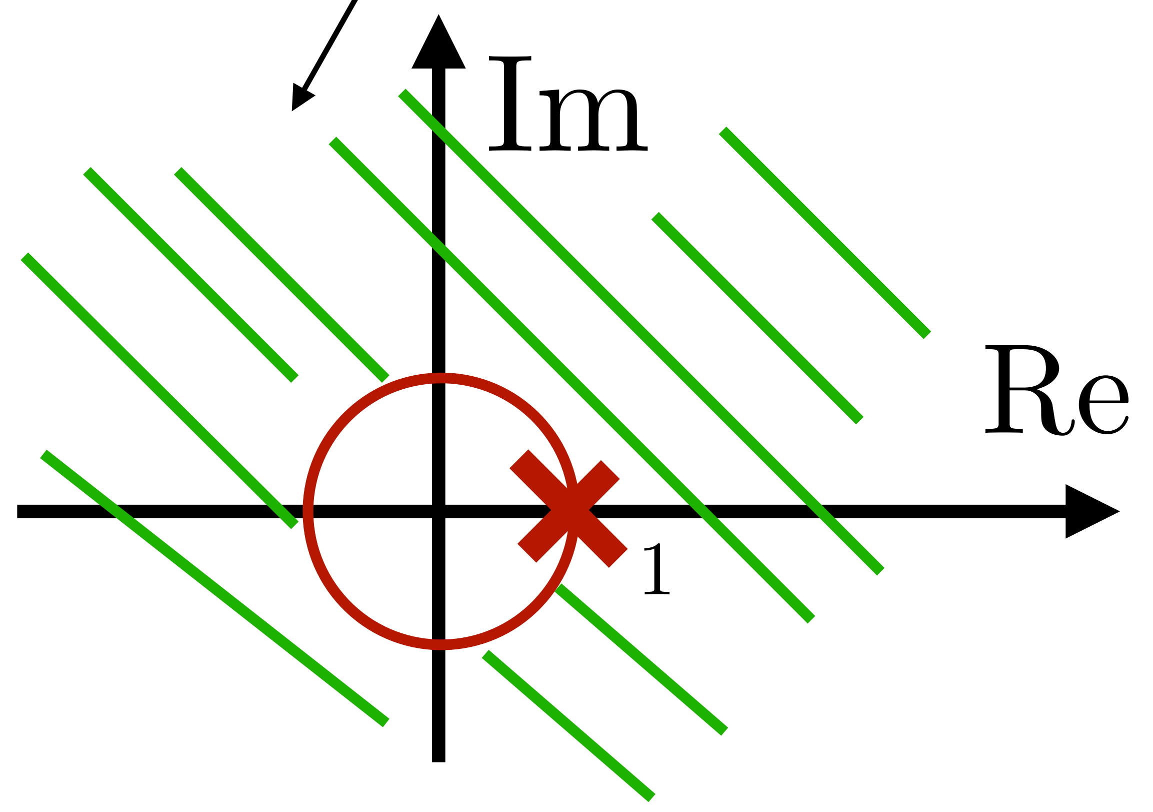
$$= \sum_{n=0}^{+\infty} z^{-n}$$

$$= \sum_{n=0}^{+\infty} (z^{-1})^{n}$$

$$= \frac{1}{1 - z^{-1}} \quad \text{ONLY IF } |z^{-1}| < 1$$

$$|z^{-1}| < 1 \Longrightarrow \left|\frac{1}{z}\right| < 1 \Longrightarrow \boxed{|z| > 1}$$

This is the ROC !!!



$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Example 7

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1} \frac{1 - r^{N_2 - N_1 + 1}}{1 - r} = \frac{r^{N_1} - r^{N_2+1}}{1 - r}$$

$$\sum_{n=0}^{N_2} r^n = \frac{1 - r^{N_2+1}}{1 - r}$$

Where we have used this one

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \qquad \boxed{\text{si } |r| < 1}$$

Example 7

(b) Zeros of X(z): $$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

The zeros are:

- 1 zero (simple - single) at z=0

Note that: $$\lim_{z \to +\infty} X(z) \approx \frac{z}{z} = 1$$

Example 7

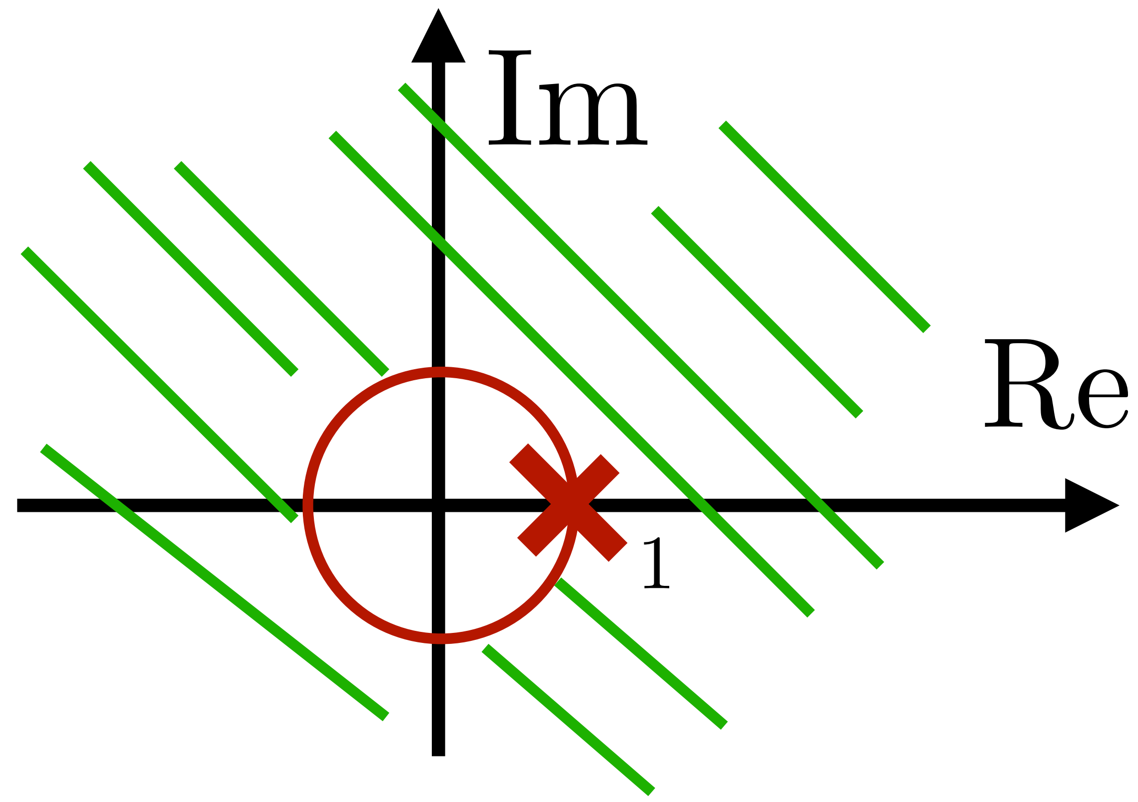
(c) Poles of X(z):

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

We have a (simple - single) pole at z=1

The the ROC is all z such that |z|>1:

$$\text{ROC: } \forall z \in \mathbb{C}: \ \text{such that } |z| > 1$$



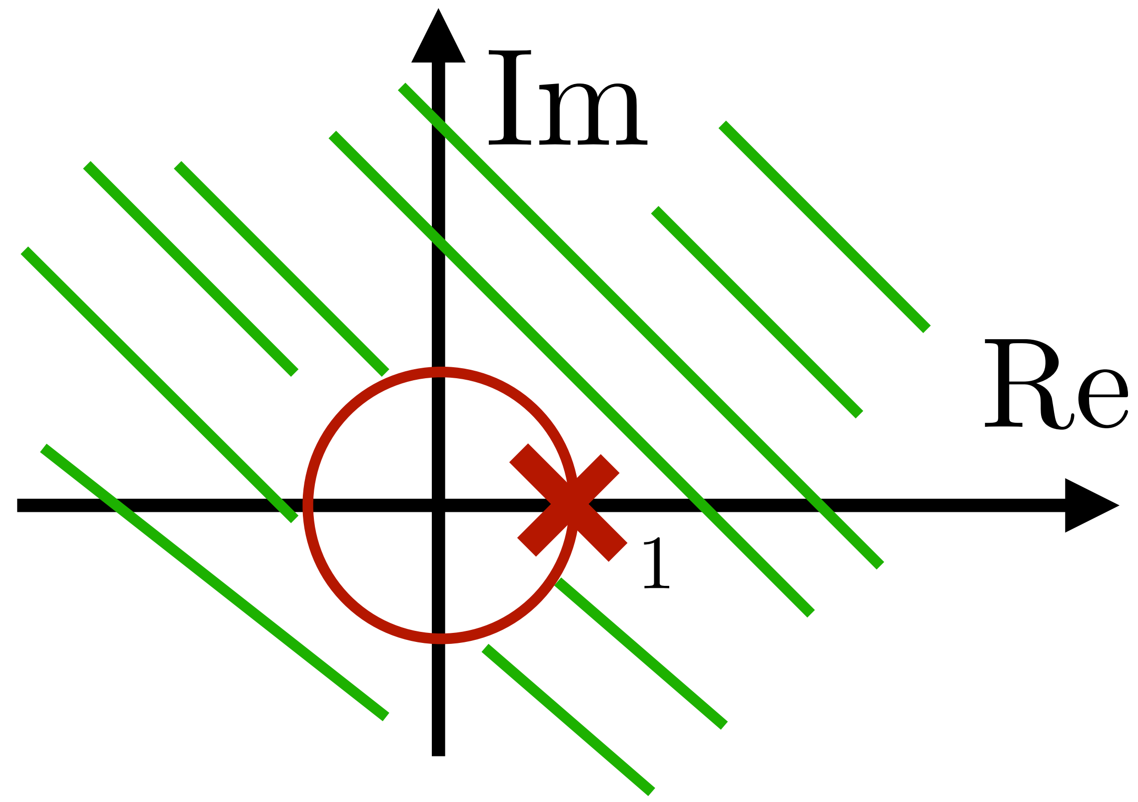
Example 7

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has infinite energy, hence the standard FT does not exist!!

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? No, then standard FT DOES NOT exists.

$$\text{ROC: } \forall z \in \mathbb{C} : \text{ such that } |z| > 1$$



The circle of radius 1 is not included in the ROC.

Example 8

Consider the following signal:

$$x[n] = u[-n]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what are the zeros of the Zeta Transform.

(c) Say what the poles of the Zeta Transform and say what is the ROC.

(d) If the stand. FT exists, compute it.

Example 8

(a) Using the direct definition:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{k=0}^{+\infty} z^{k}$$

$$k = -n \rightarrow n = -k$$

Example 8

We have to recall:

$$\sum_{n=N_1}^{N_2} r^n = r^{N_1}\frac{1-r^{N_2-N_1+1}}{1-r} = \frac{r^{N_1}-r^{N_2+1}}{1-r}$$

$$\sum_{n=0}^{N_2} r^n = \frac{1-r^{N_2+1}}{1-r}$$

We will use this one

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \boxed{\text{si } |r|<1}$$

Example 8

Then:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

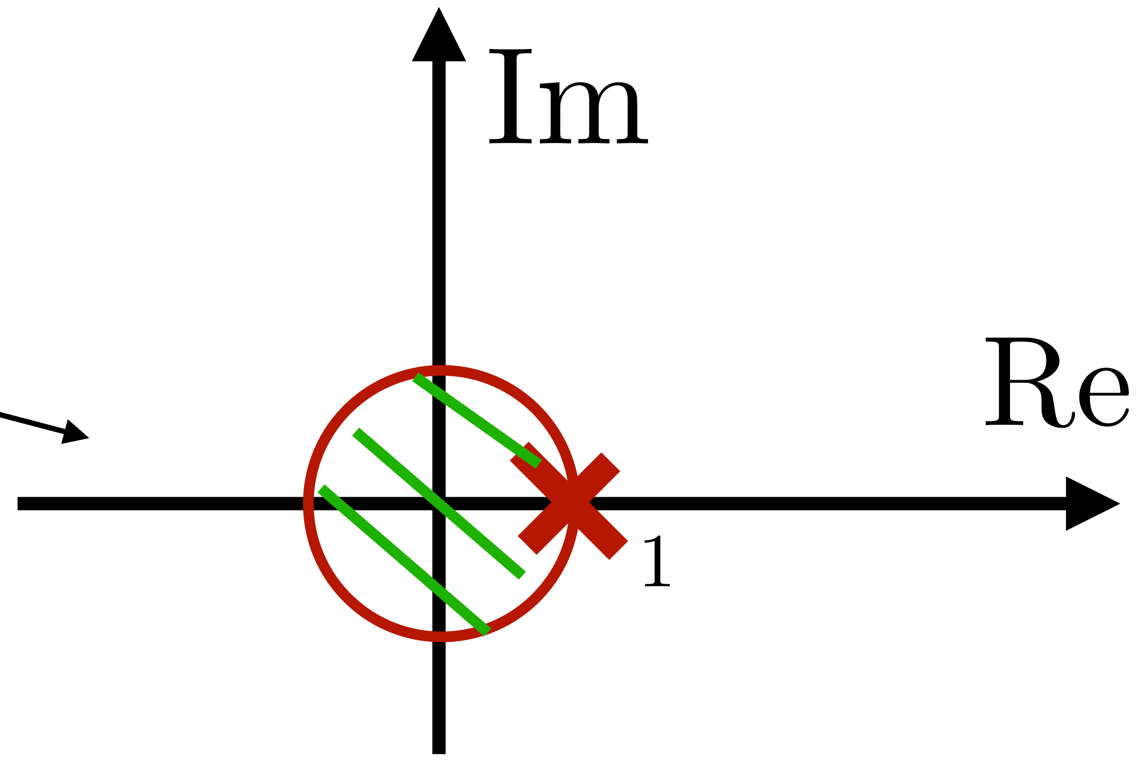
$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{k=0}^{+\infty} z^{k}$$

$$= \frac{1}{1-z}$$

ONLY IF $|z| < 1$

This is the ROC !!!



Example 8

(b) Zeros of X(z):

$$X(z) = \frac{1}{1-z}$$

The zeros are:

- **1 zero (simple - single) at z=Infinity**

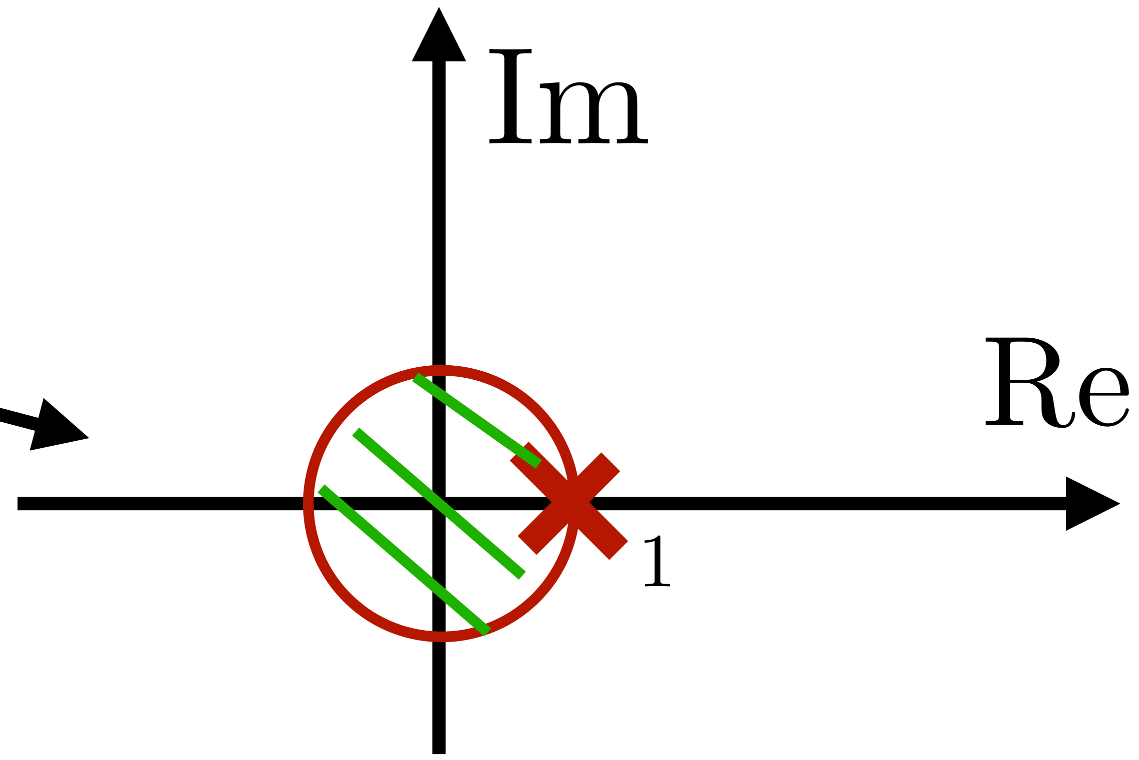
Example 8

(c) Poles of X(z): $$X(z) = \frac{1}{1-z}$$

We have a (simple - single) pole at z=-1

The the ROC is all z such that |z|<1:

$$\text{ROC: } \forall z \in \mathbb{C} : \text{ such that } |z| < 1$$



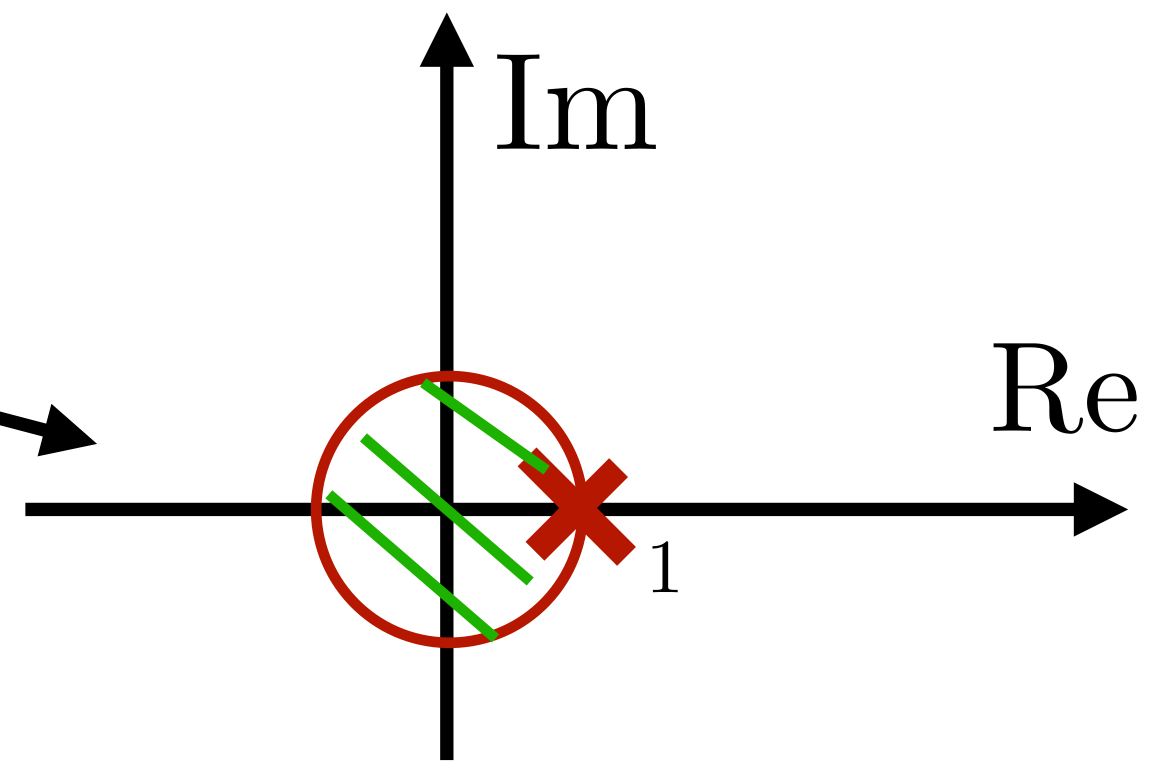
Example 8

(d) We can make two considerations (we have two way of proceeding):

- We can observe that the signal has infinite energy, hence the standard FT does not exist!!

- The second way is to have a look to the ROC: does the ROC include the circle of radius 1? No, then standard FT DOES NOT exists.

$$\text{ROC: } \forall z \in \mathbb{C} : \ \text{such that } |z| < 1$$



The circle of radius 1 is not included in the ROC.

Example 9

Consider the following signal:

$$x[n] = \sum_{j=0}^{L-1} a_n \delta[n-j]$$

(a) Compute the Zeta Transform: $X(z) = ?$

(b) Say what the poles of the Zeta Transform and say what is the ROC.

Example 9

ZT of finite length signals

For instance: $$x[n] = \sum_{j=0}^{L-1} a_n \delta[n-j]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{L-1} x[n]z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + ... + x[L-1]z^{-(L-1)}$$

$$= \frac{x[0]z^{L-1} + x[1]z^{L-2} + x[2]z^{L-3} + ... + x[L-1]}{z^{L-1}}$$

ROC= ? ➔

Example 9

ROCs of ZT
of finite length signals

If x[n] has finite length, the ROC is all the complex plane except POSSIBLY the points z=0 and/or z=Infinity.

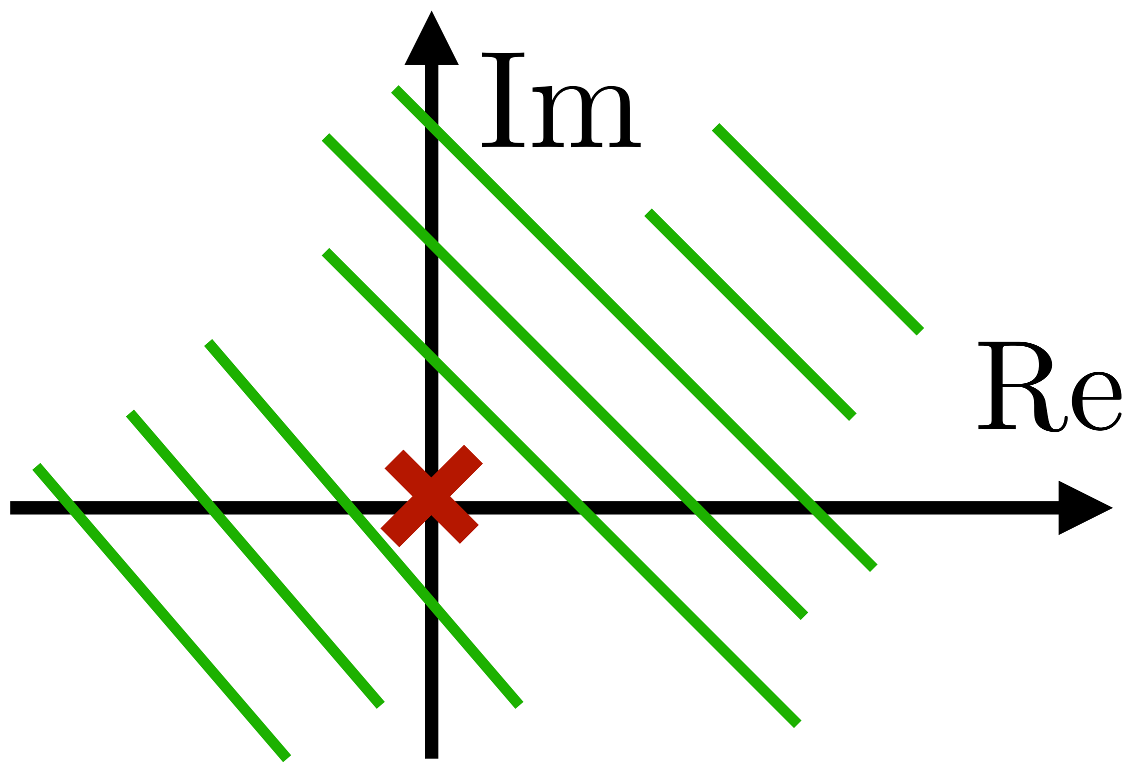
The ROC is all the complex plane just for x[n]=Delta[n].

Example 9

in our specific case, the poles are just at z=0, since at z=Infinity, we have:

$$\lim_{z \to +\infty} X(z) \approx \frac{z^{L-1}}{z^{L-1}} = 1$$

in fact, it is a right-sided sequence-signal !!



Questions?