

Discrete Fourier Transform and FFT

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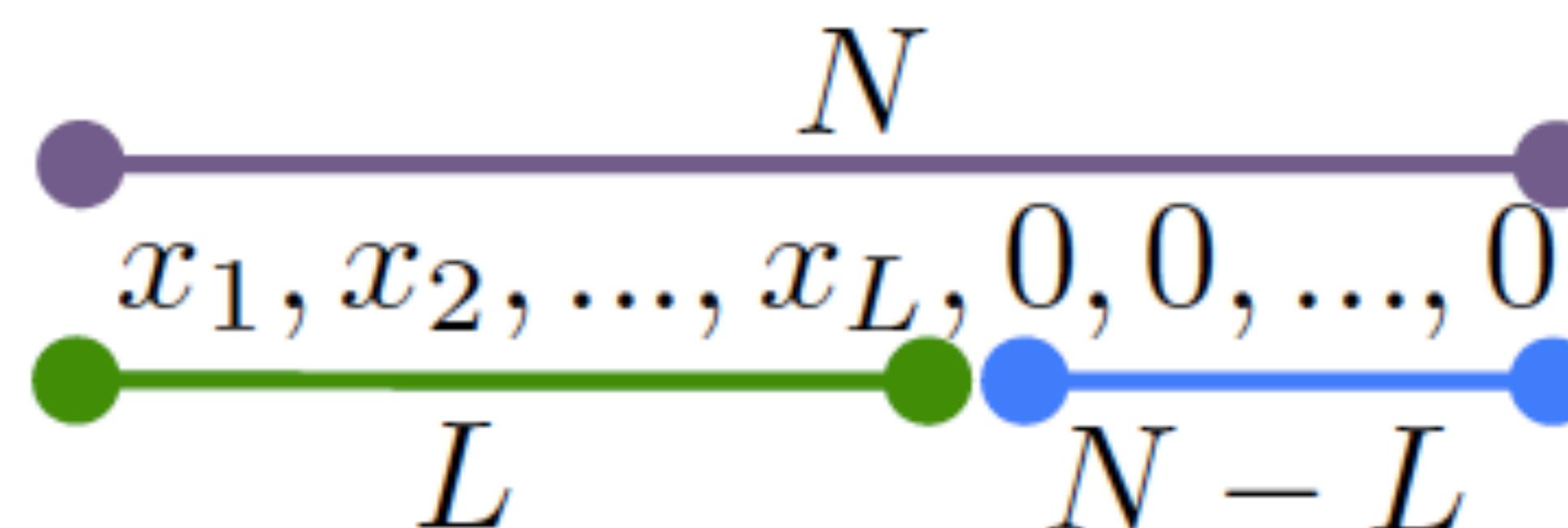
Interpretation as a *periodic signal*

The easiest (computationally speaking) is to interpret the sequence of values as one period of a *periodic signal* of period $N \geq L$

$$\boxed{N \geq L}$$

(we will consider other possibilities in other slides)

- We said $N \geq L$ since we can always fill with $N-L$ zeros (adding more zeros) our sequence of values:



Analysis Equation of DFT

$$X_N[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi}{N} kn}, \quad k = 0, \dots, N-1$$

Synthesis equation of DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N[k] \cdot e^{+j \frac{2\pi}{N} nk}, \quad n = 0, \dots, N-1$$

Example

Then we have if we consider $N=L=3$ then $N-1=2$:

$$X_N[k] = \sum_{n=0}^2 x[n] \cdot e^{-j \frac{2\pi}{N} kn} \quad k = 0, 1, N-1 = 2$$

$$X_N[k] = 3 - 5e^{-jk \frac{2\pi}{N}} + 1.5e^{-j2k \frac{2\pi}{N}}$$

$n = 0$ $n = 1$ $n = 2$

Example

Then we have if we consider $N=L=3$ then $N-1=2$:

$$X_N[k] = 3 - 5e^{-jk\frac{2\pi}{N}} + 1.5e^{-j2k\frac{2\pi}{N}}$$

The equation is shown in a light green box. Below it, three orange arrows point upwards from the labels $n = 0$, $n = 1$, and $n = 2$ to the terms $5e^{-jk\frac{2\pi}{N}}$, $e^{-j2k\frac{2\pi}{N}}$, and $1.5e^{-j2k\frac{2\pi}{N}}$ respectively.

$$X_3[k] = 3 - 5e^{-jk\frac{2\pi}{3}} + 1.5e^{-j2k\frac{2\pi}{3}}$$

Example

$$X_3[k] = 3 - 5e^{-jk\frac{2\pi}{3}} + 1.5e^{-j2k\frac{2\pi}{3}}$$



$$k = 0, 1, N - 1 = 2$$

OUTPUT of the FFT:

$$\begin{aligned} X_3[0] &= 3 - 5e^{-j0\frac{2\pi}{3}} + 1.5e^{-j2 \cdot 0 \frac{2\pi}{3}} \\ &= 3 - 5 + 1.5 = -0.5 \end{aligned}$$

Example

$$X_3[k] = 3 - 5e^{-jk\frac{2\pi}{3}} + 1.5e^{-j2k\frac{2\pi}{3}}$$



$$k = 0, 1, N - 1 = 2$$

OUTPUT of the FFT:

$$\begin{aligned} X_3[1] &= 3 - 5e^{-j1\frac{2\pi}{3}} + 1.5e^{-j2 \cdot 1 \frac{2\pi}{3}} \\ &= 3 - 5e^{-j\frac{2\pi}{3}} + 1.5e^{-j\frac{4\pi}{3}} \end{aligned}$$

```
>> 3-5*exp(-i*(2*pi)/3)+1.5*exp(-i*(4*pi)/3)
```

```
ans =
```

```
4.7500 + 5.6292i
```

Example

$$X_3[k] = 3 - 5e^{-jk\frac{2\pi}{3}} + 1.5e^{-j2k\frac{2\pi}{3}}$$



$$k = 0, 1, N - 1 = 2$$

OUTPUT of the FFT:

$$X_3[2] = 3 - 5e^{-j\frac{4\pi}{3}} + 1.5e^{-j\frac{8\pi}{3}}$$

```
>> 3-5*exp(-i*(4*pi)/3)+1.5*exp(-i*(8*pi)/3)
```

```
ans =
```

```
4.7500 - 5.6292i
```

Example

OUTPUT of the FFT:

```
>> fft([3 -5 1.5])  
  
ans =  
  
-0.5000 + 0.0000i    4.7500 + 5.6292i    4.7500 - 5.6292i
```

Example

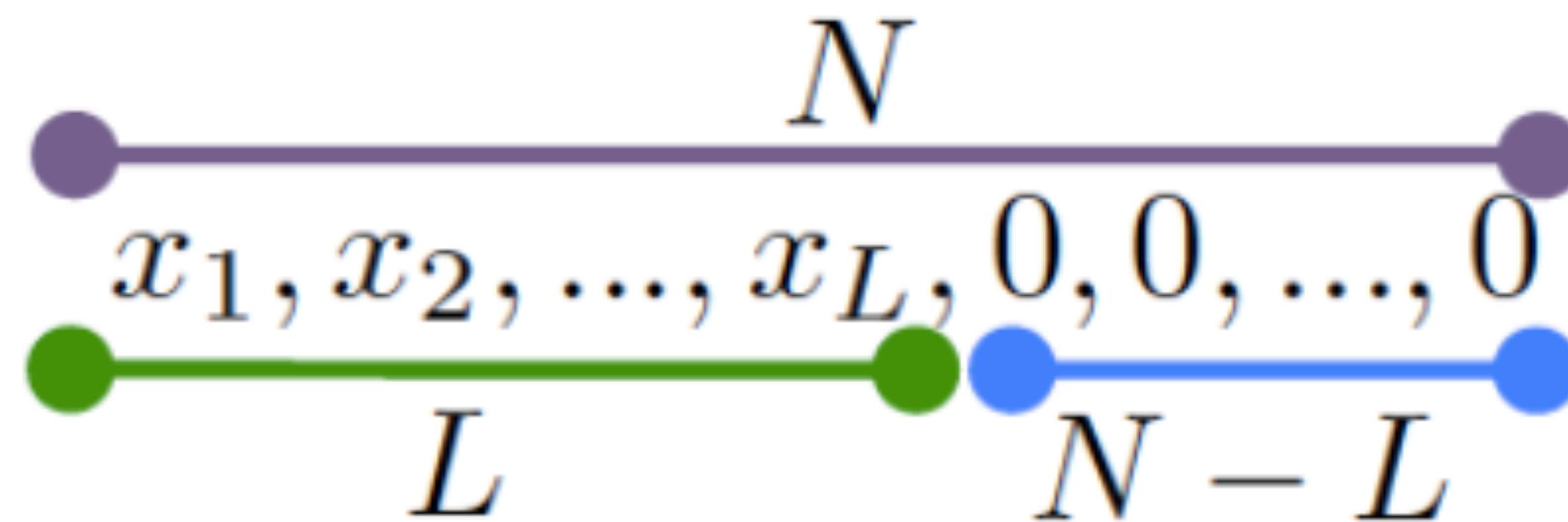
Let consider the following signal:

$$x[0] = 3, x[1] = -5, x[2] = 1.5, \quad x[n] = 0 \quad \text{for the rest of } n$$

Then we have: $L = 3$

Consider: N=4

➤ We said $N \geq L$ since we can always fill with $N-L$ zeros
(adding more zeros) our sequence of values:



$$X_N[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi}{N} kn} \quad k = 0, 1, 2, \dots, N-1$$



$$X_N[k] = x[0] + x[1] \cdot e^{-j \frac{2\pi}{N} k} + x[2] \cdot e^{-j \frac{4\pi}{N} k} + \dots + x[L-1] \cdot e^{-j \frac{2(L-1)\pi}{N} k} + 0 + 0 + 0\dots$$



$$X_N[k] = 3 - 5e^{-jk \frac{2\pi}{N}} + 1.5e^{-jk \frac{4\pi}{N}}$$

$$X_4[k] = 3 - 5e^{-jk \frac{2\pi}{4}} + 1.5e^{-jk \frac{4\pi}{4}}$$

$$X_4[k] = 3 - 5e^{-jk \frac{\pi}{2}} + 1.5e^{-jk\pi}$$

$$X_4[k] = 3 - 5e^{-jk\frac{\pi}{2}} + 1.5e^{-jk\pi}$$

```
>> fft([3 -5 1.5],4)
```

```
ans =
```

```
Columns 1 through 3
```

```
-0.5000 + 0.0000i  1.5000 + 5.0000i  9.5000 + 0.0000i
```

```
Column 4
```

```
1.5000 - 5.0000i|
```

Questions?