

Topic 3 - Part 3: Fourier Series

Linear systems and circuit applications

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Based also on Professor Óscar Barquero Perez, Andrés Martínez and José Luis Rojo's slides

Transformations for signal in continuous time

For Periodic signals

For non-periodic signals

Fourier Series (FS) | **Stand. Fourier Transform (FT)** | **Laplace Transform (FT)**

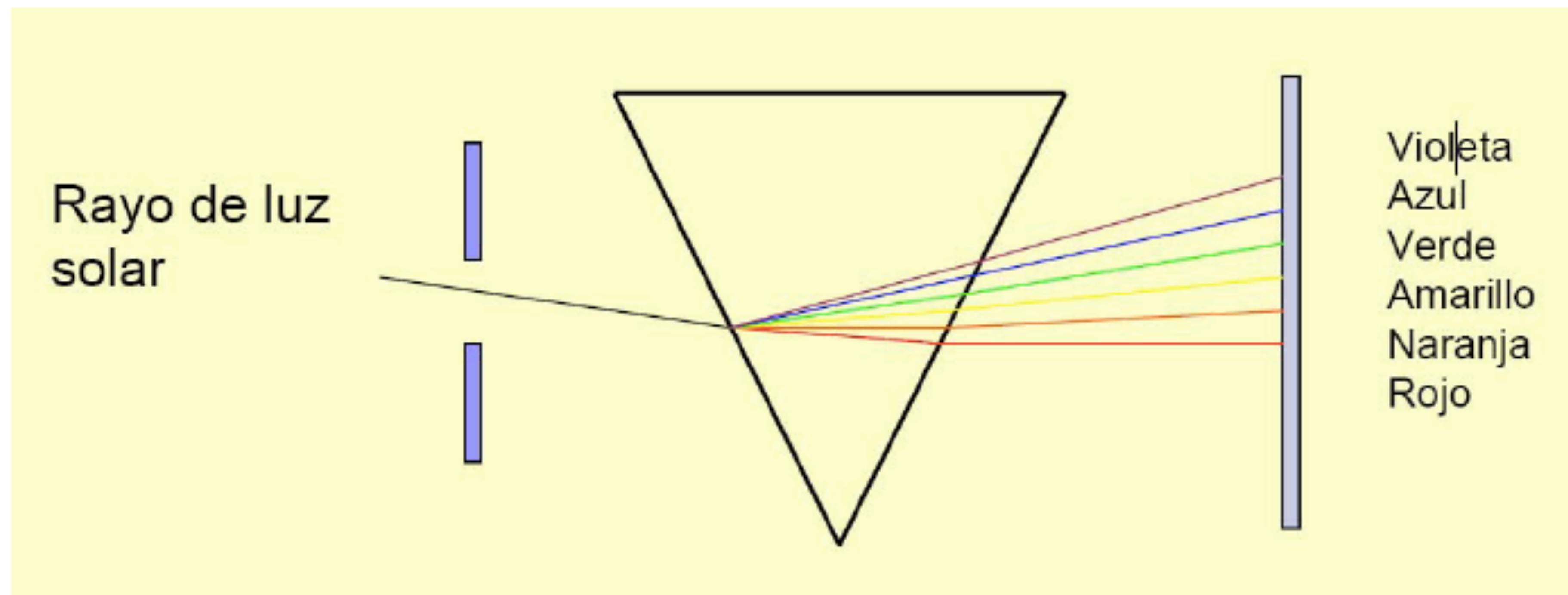
also for some
Signals with
Infinite Energy

Generalized
Fourier Transform
(GFT)

*Mathematically, it is not
completely valid... or we need
other definition of Fourier
Transformation....*

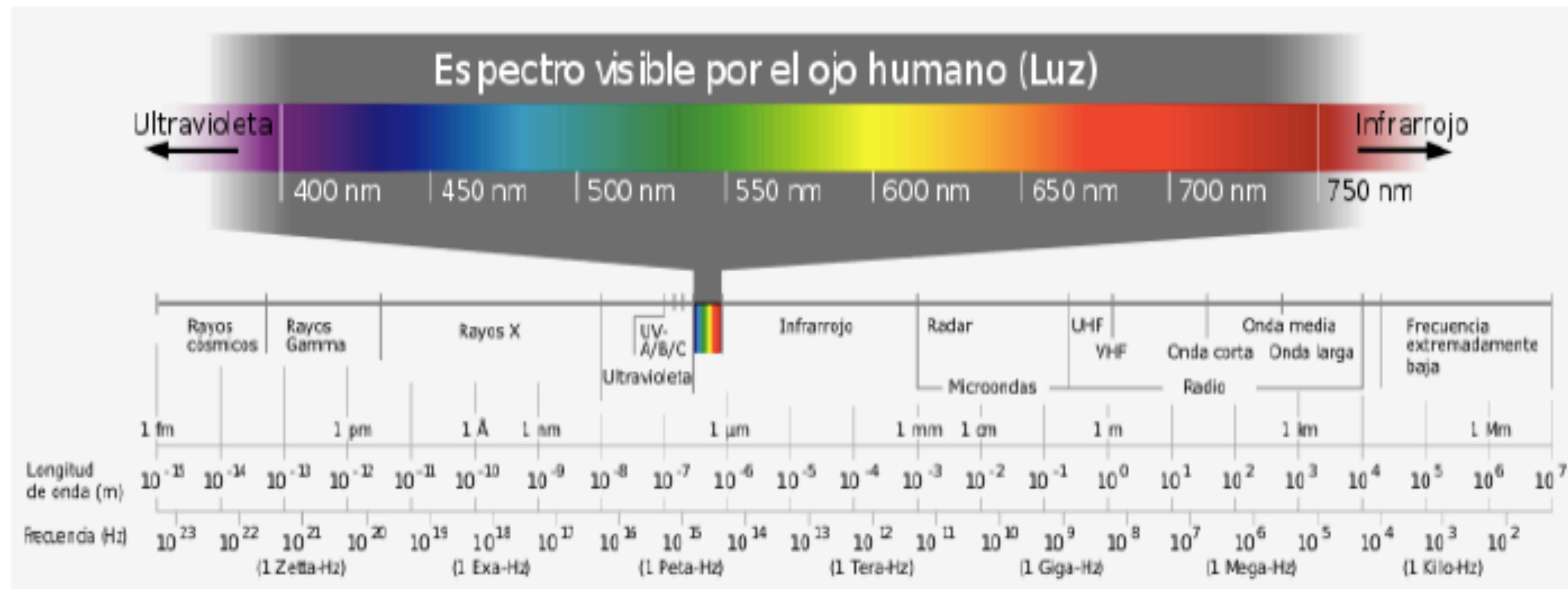
Example: “dividing” a signal - “discovering pieces”

- Ejemplo de descomposición de Fourier:
 - ❖ Luz blanca que atraviesa un prisma



Example: “dividing” a signal - frequency

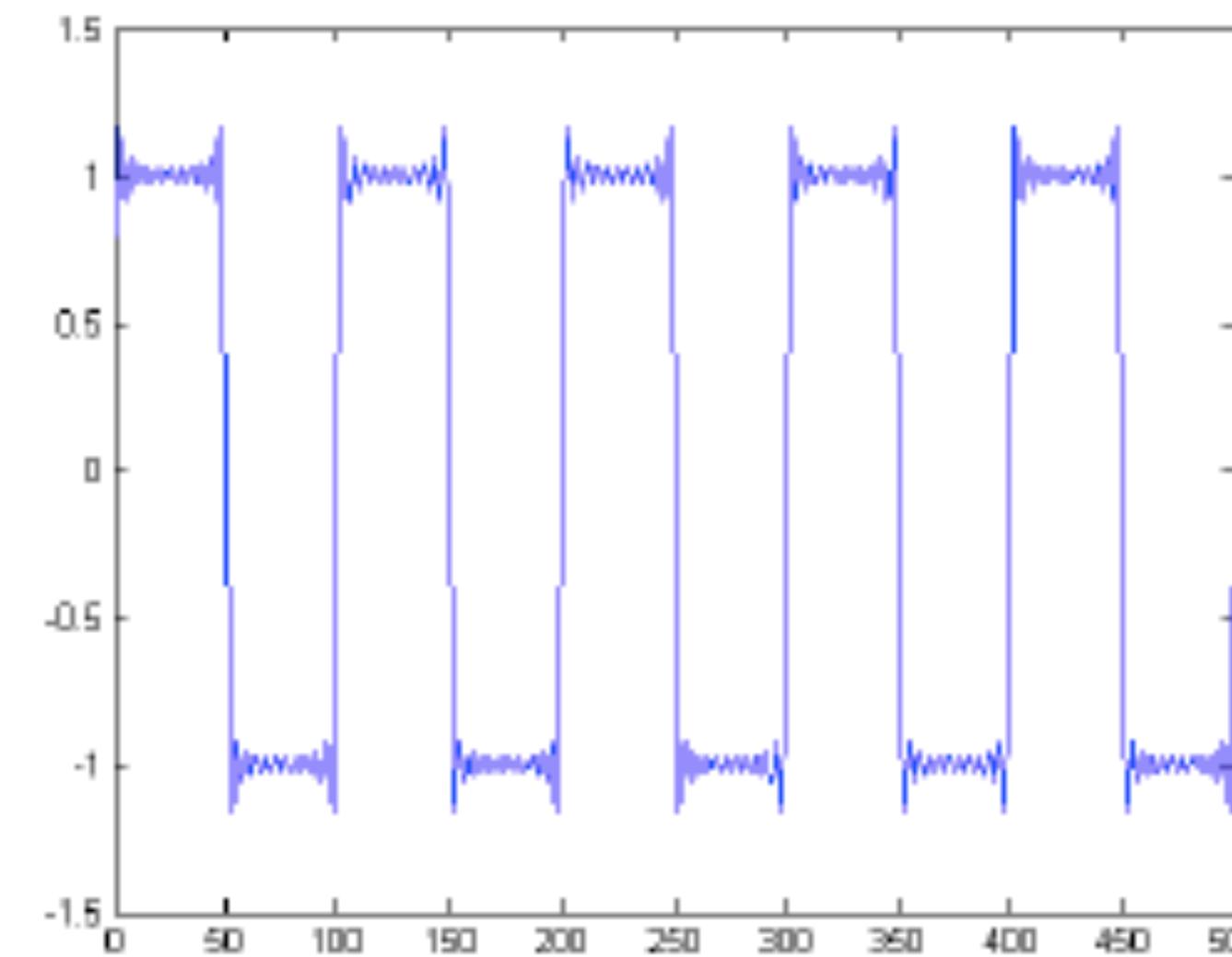
Espectro Electromagnético



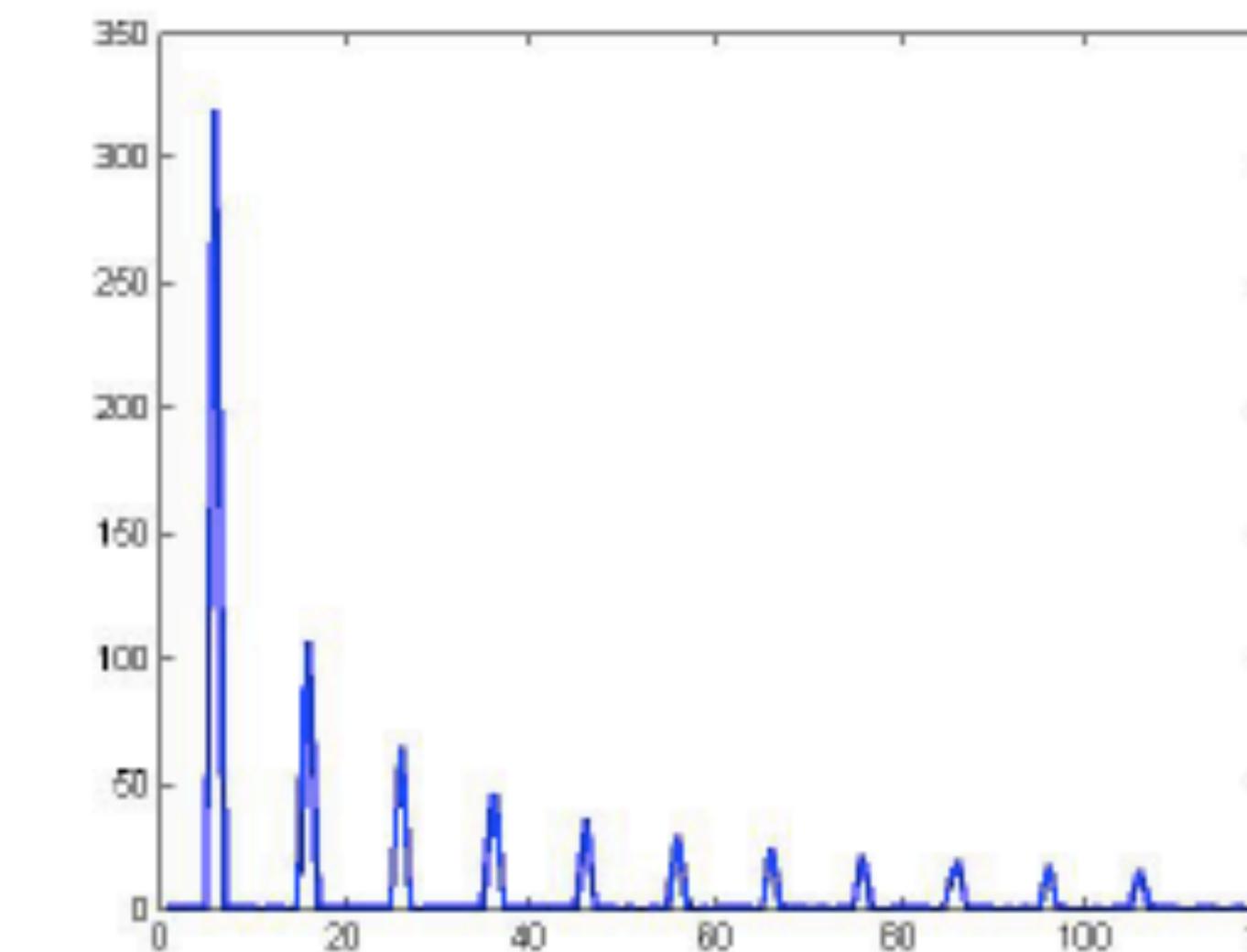
frequency = reciprocal of the wave length

“Dividing-expressing” a signal in components of different frequencies

- ¿Qué hay detrás de una señal? ...
 - ❖ Diversas componentes de frecuencia y amplitud



Dominio del tiempo
(continuo o discreto)



Dominio de la frecuencia

¿Cómo se realiza este análisis en frecuencia?

Next?

- Fourier Series (FS)
- Only for periodic signals

For a periodic signal? transformed domain?

- Historically, the Fourier Series was the first tool introduced...

Periodic signals

- A periodic signal satisfies:

$$x(t) = x(t + T_0)$$

$T_0 \implies$ fundamental period

$$\omega_0 = \frac{2\pi}{T_0} \implies \text{fundamental frequency}$$

$$f_0 = \frac{1}{T_0}, \quad \omega_0 = 2\pi f_0$$

Periodic signals

- Clearly we have also:

$$x(t) = x(t - T_0)$$

$$x(t) = x(t + 2T_0)$$

$$x(t) = x(t + 3T_0)$$

$$x(t) = x(t + 4T_0)$$

...

Periodic signals

- Now consider a signal $x(t)$ with fundamental period $T_0/2$:

$$x(t) = x\left(t + \frac{T_0}{2}\right)$$

$$x(t) = x\left(t + 2\frac{T_0}{2}\right) = x(t + T_0)$$

...

- Then $x(t)$ fulfills also: $x(t) = x(t + T_0)$

Periodic signals

- This is true for any signal $x(t)$ with fundamental period T_0/k ,
- where k is an integer (positive or negative):

$$x(t) = x\left(t + \frac{T_0}{k}\right)$$

$$x(t) = x\left(t + k \frac{T_0}{k}\right) = x(t + T_0)$$

$$x(t) = x(t + T_0)$$

Periodic signals

- Then all the signals with fundamental period T_0/k are also periodic with period T_0 .
- Then the signals also contains the frequencies which are multiple of the fundamental frequency:

$$\omega_k = k \frac{2\pi}{T_0} = k\omega_0$$

conflicto de notación con $k=0$ (que es un valor posible)...

con $k=0$ es la frecuencia nula, pero no es la frecuencia fundamental ω_0

Fourier Series

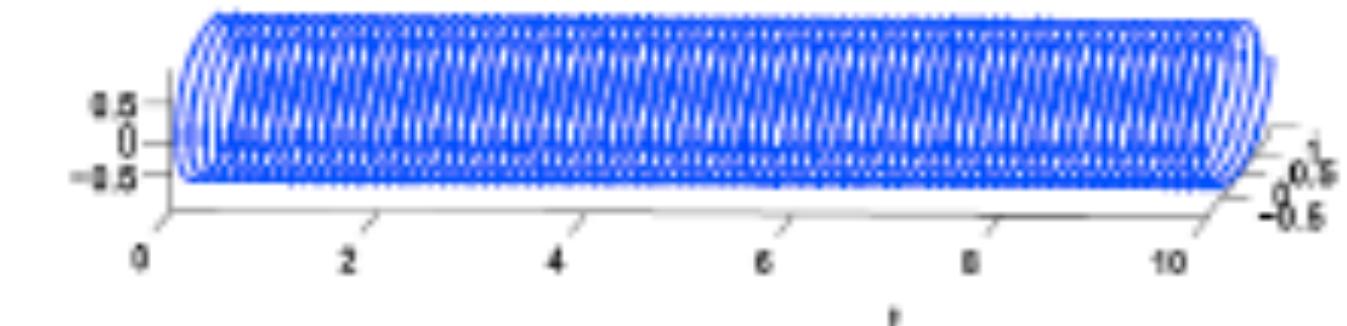
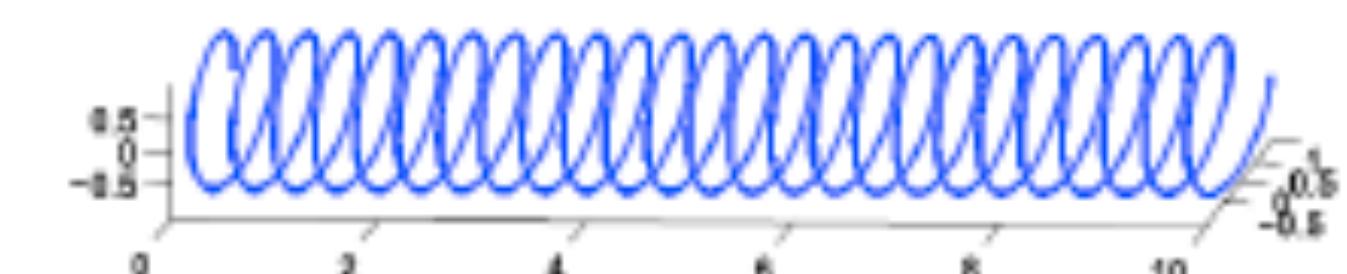
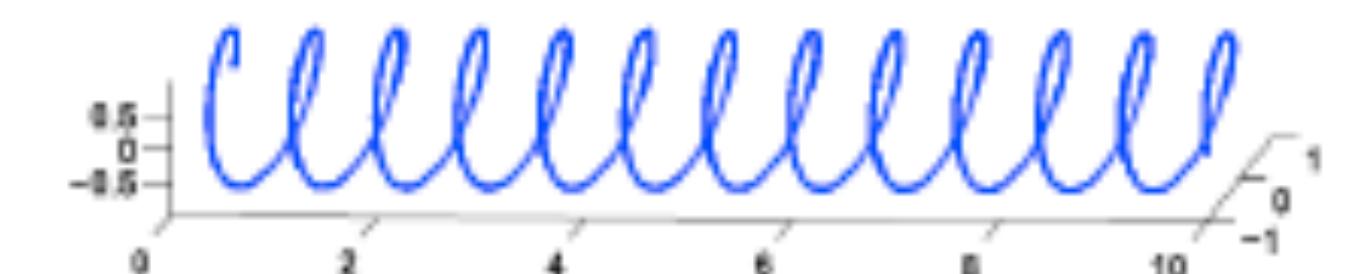
Fourier series representation

- Jean Baptiste J Fourier (advisor and soldier with Napoleon, mathematician and politician) proved in 1807 that any periodic signal with fundamental period T_0 can be represented as a linear combination (weighted sum) of complex exponential functions.

- The set of harmonically related complex exponentials is defined as:

$$\phi_k(t) = e^{jk\frac{2\pi}{T_0}t}, \text{ con } k = 0, \pm 1, \pm 2, \dots$$

- With fundamental periods: $T_0, \frac{T_0}{2}, \frac{T_0}{3}, \dots$
- And frequencies: $f_0, 2f_0, 3f_0, \dots$



Fourier Series

- Then, if $x(t) = x(t + T_0)$, it may be represented using Fourier series as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier Series

- Definition of the Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j k \omega_0 t}$$

- It is a series then we have to consider the convergence.
- it is a decomposition of the signal $x(t)$ with respect to *the periodic bases*.

Fourier Series

- Definition of the Fourier series (**synthesis equation**):

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j k \omega_0 t}$$

- **a_k: complex coefficients**
- **k integer variable**

Fourier Series

- Definition of the Fourier series :

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j k \omega_0 t}$$

- **Synthesis equation:** from a_k to $x(t)$

$$a_k \implies x(t)$$

- we need to know a_k

Fourier Series: frequency information?

- Frequency information is contained in the a_k 's

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

- INTEGRAL IN A PERIOD** - from 0 to T_0 or $-T_0/2$ to $T_0/2$, for instance.
- a_k : complex coefficients
- k integer variable

Fourier Series: frequency information?

- Definition of the a_k (it is possible to prove it):

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$

- the definition is a definite integral in a period (**analysis equation**)

Fourier Series: frequency information?

- Relationship with the frequencies contained in the signal:

conflicto de notación con k=0 (que es un valor posible)...

con k=0 es la frecuencia nula, pero no es la frecuencia fundamental ω_0

$$a_k \xrightarrow{\text{ }} \omega_k = k\omega_0$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j k \omega_0 t} dt$$

Which frequencies are into a periodic signal?

- Does a periodic signal contain all the frequencies?
- **NO!! only the multiple of the fundamental frequency.**

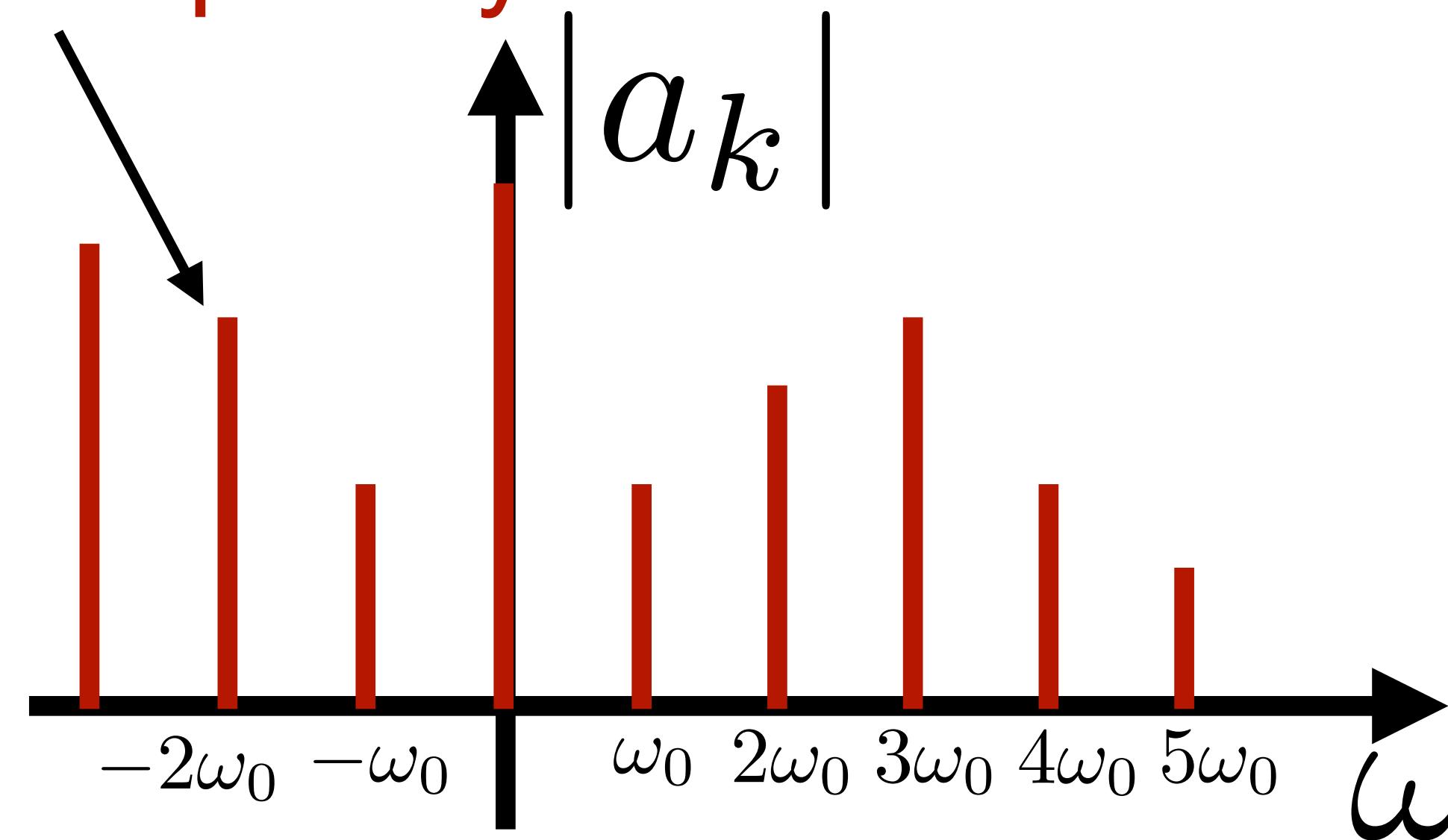
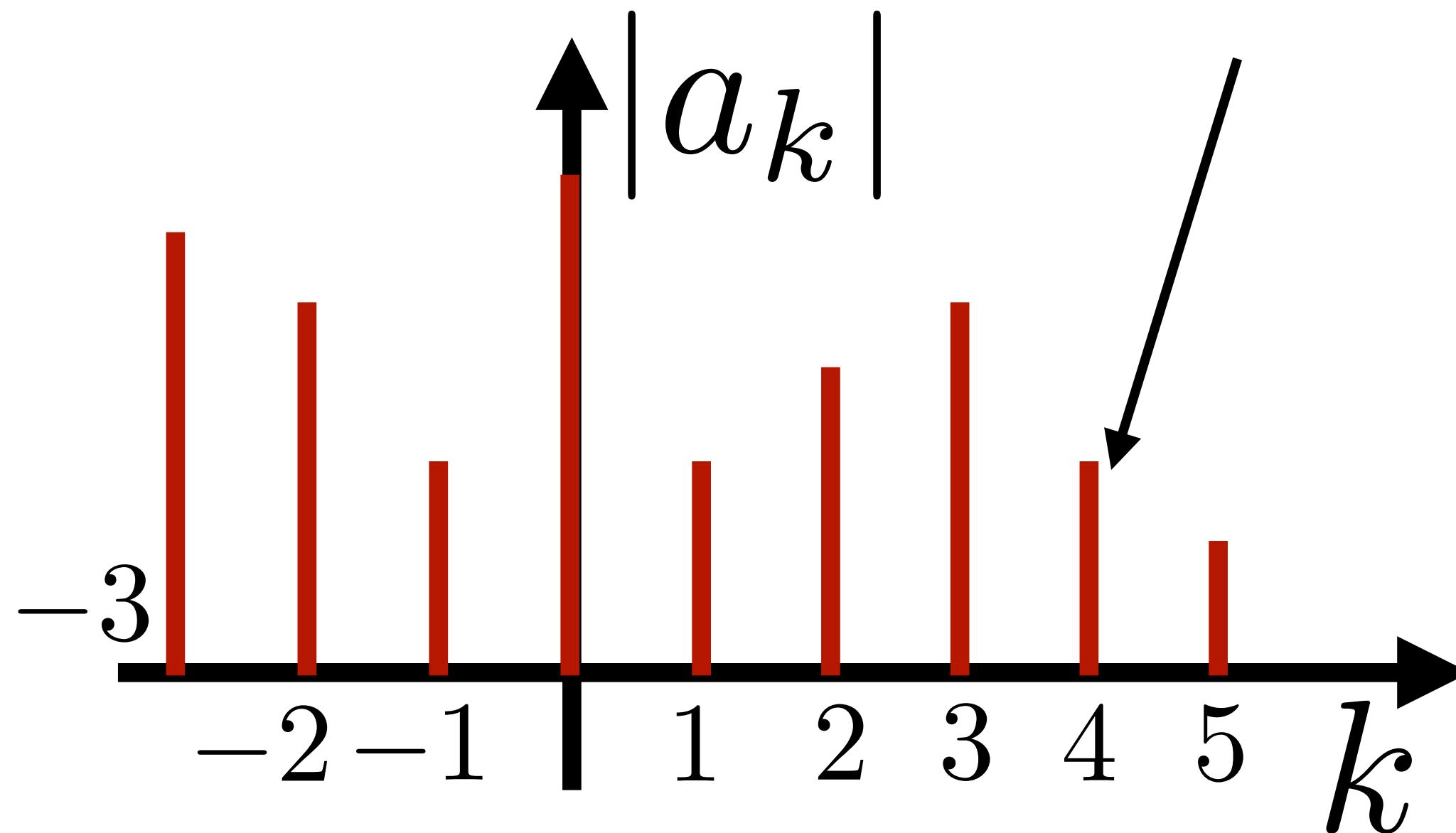
$$a_k \implies \omega_k = k\omega_0$$

conflicto de notación con $k=0$ (que es un valor posible)...
con $k=0$ es la frecuencia nula, pero no es la frecuencia fundamental ω_0

Which frequencies are into a periodic signal?

- The coefficients a_k 's are complex numbers

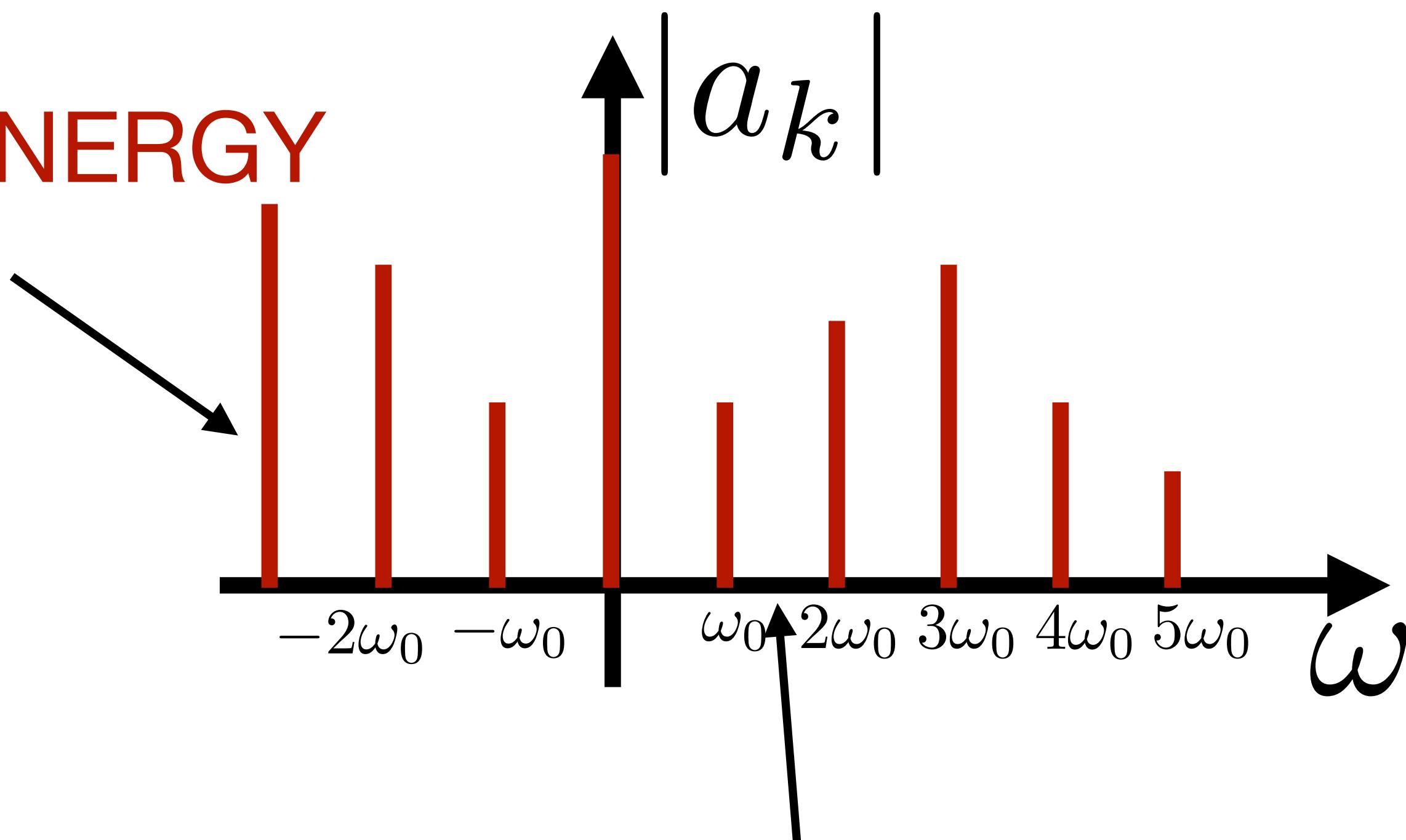
- (square root of) ENERGY
in each frequency



Which frequencies are into a periodic signal?

- Important observation:

- (square root of) ENERGY
in each frequency



- In the middle is not defined...
these frequencies are not contained

Now a brief summary

- Summary for the Fourier series representation for continuous-time periodic signals:

- **Synthesis equation:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- **Analysis equation:**

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

- The integral can be done in each interval of length T_0 (the period)

Observation: a_0

- Note that:

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

- it is the averaged value of the signal (“valor medio”) in one period.

Other forms of seeing the FS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j \textcolor{red}{k} \omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k [\cos(k\omega_0 t) + j \sin(k\omega_0 t)]$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cos(k\omega_0 t) + j \sum_{k=-\infty}^{+\infty} a_k \sin(k\omega_0 t)$$

Other forms of seeing the FS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j k \omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \cos(k \omega_0 t) + \sum_{k=-\infty}^{+\infty} b_k \sin(k \omega_0 t)$$

with $b_k = +j a_k$

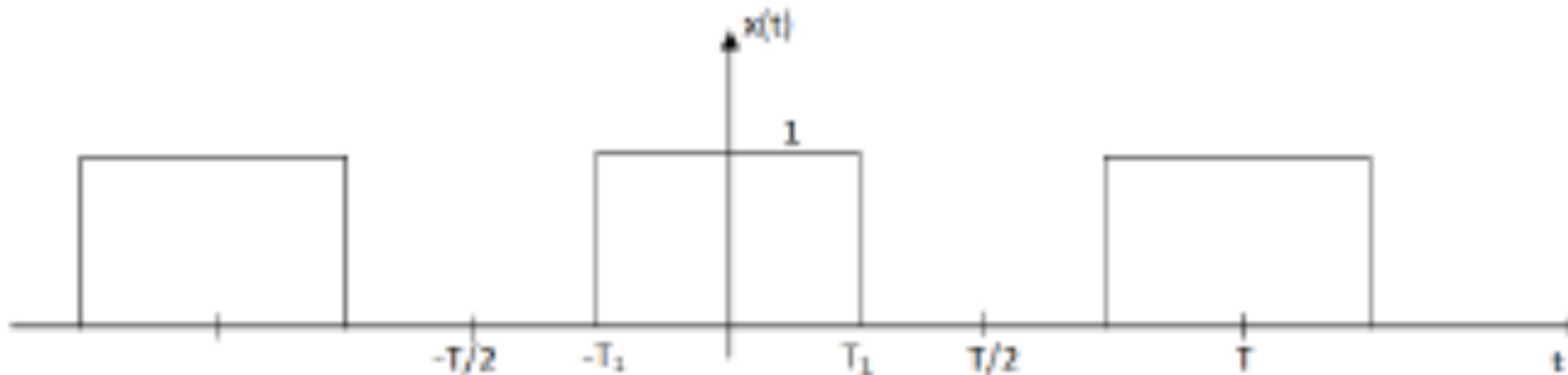
Recall: a_k are complex numbers, in general

Other forms of seeing the FS

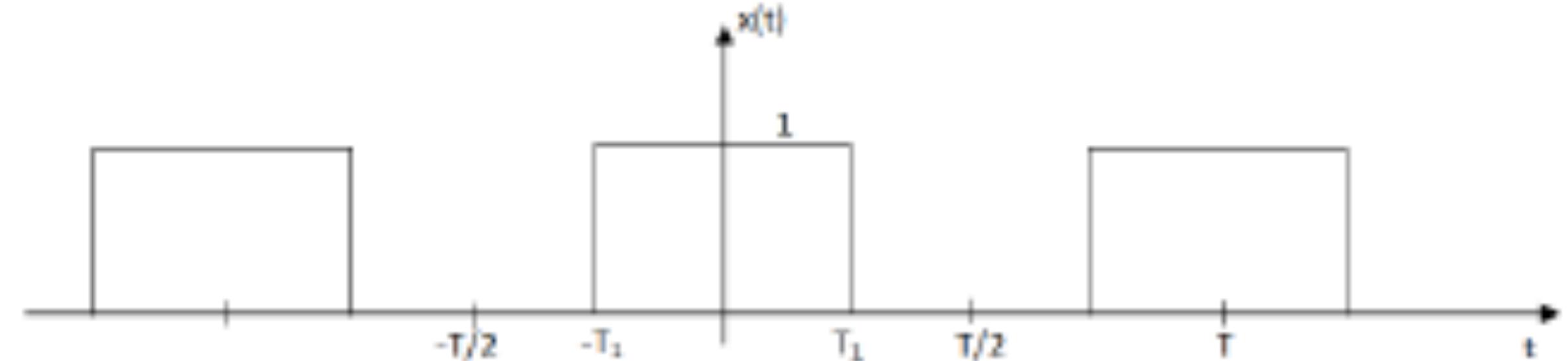
- There are many other mathematical forms: depending if the signal is real, if the signal is odd or even, or odd and real, or even and real etc... (you will see some examples later)
- But it is the same “mathematical tool”
- (se puede escribir de diferentes formas, basta ser coherentes con la ecuación de análisis y síntesis, etc.)

Important example

$$x(t) = \begin{cases} 1, & \text{si } |t| < T_1 \\ 0, & \text{si } T_1 < |t| < T/2 \end{cases}$$



Important example



- As $x(t)$ is periodic it can be represented using Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
- Coefficient calculation:

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} 1 e^{-jk\omega_0 t} dt =_{(k \neq 0)}$$

$$= \frac{1}{T} \frac{-1}{jk\omega_0} [e^{-jk\omega_0 t}]_{-T_1}^{T_1} = \frac{-1}{jk\omega_0 T} [e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}] = \dots = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

- IMP: the notation for the period here is T , not T_0

Important example

- we give more details
- we come back the notation for the period T_0

$$a_k = \frac{-1}{jk\omega_0 T_0} [e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}]$$



$$\omega_0 = \frac{2\pi}{T_0}$$

$$\sin(k\omega_0 T_1) = \frac{1}{2j} [e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}]$$

$$-2j \sin(k\omega_0 T_1) = -e^{jk\omega_0 T_1} + e^{-jk\omega_0 T_1}$$

Important example

- we give more details

$$a_k = \frac{-1}{jk\omega_0 T_0} [e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}]$$

replacing inside:

$$a_k = \frac{-1}{jk2\pi} [-2j \sin(k\omega_0 T_1)]$$

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

Important example

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

- in this example, the a_k are real: this is due to $x(t)$ is real and even...
- in $k=0$, we have an indeterminate form (0/0), we can solve in this way (again the notation for the period here is T , not T_0):
- For $k = 0$, we calculate the coefficient independently:

$$a_0 = \frac{1}{T} \int_T x(t) e^{j0\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} 1 dt = \frac{2T_1}{T}$$

Important example: final solution

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$a_0 = \frac{2T_1}{T_0}$$

Convergence

- Consider a **truncated Fourier Series:**

$$x(t) \approx \sum_{k=-N}^{N} a_k e^{+jk\omega_0 t}$$

when N goes to infinity, we recover the Fourier Series.

Convergence

when N goes to infinity, we recover the Fourier Series:

$$x(t) \approx a_{-N} e^{-jN\omega_0 t} + \dots a_{-1} e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \dots + a_N e^{jN\omega_0 t}$$

$2N+1$ is the number of total components in the sum!

$$x_N(t) = \sum_{k=-N}^N a_k e^{+jk\omega_0 t}$$

Convergence

- We can define the “**error signal**” - error in approximation as:

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^N a_k e^{jk\omega_0 t}$$

- y definimos el error (en un periodo) como:

$$E_N = \int_{T_0} |e_N(t)|^2 dt$$

Convergence

- **For a fixed N:** It is possible to prove that the best choice (i.e., that minimizes) of the coefficient a_k is

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j k \omega_0 t} dt$$

- i.e., the coefficients of the Fourier Series (that we have already defined and studied).
- **Increasing N, with the a_k above:** the error E_N decreases as N grows, i.e.,

$$E_N \rightarrow 0 \quad \text{as} \quad N \rightarrow +\infty$$

Convergence

A nice summary in Spanish:

Entonces, si $x(t)$ tiene una representación en serie de Fourier, la mejor aproximación usando sólo un número finito de exponenciales complejas relacionadas armónicamente se obtiene truncando la serie de Fourier al número de términos deseado. Conforme N se incrementa, se suman nuevos términos y E_N disminuye. De hecho, si $x(t)$ tiene una representación en serie de Fourier, entonces el límite de E_N cuando $N \rightarrow \infty$ es cero.

Convergence: example

think to a periodic "rectangular" signal

Convergence example

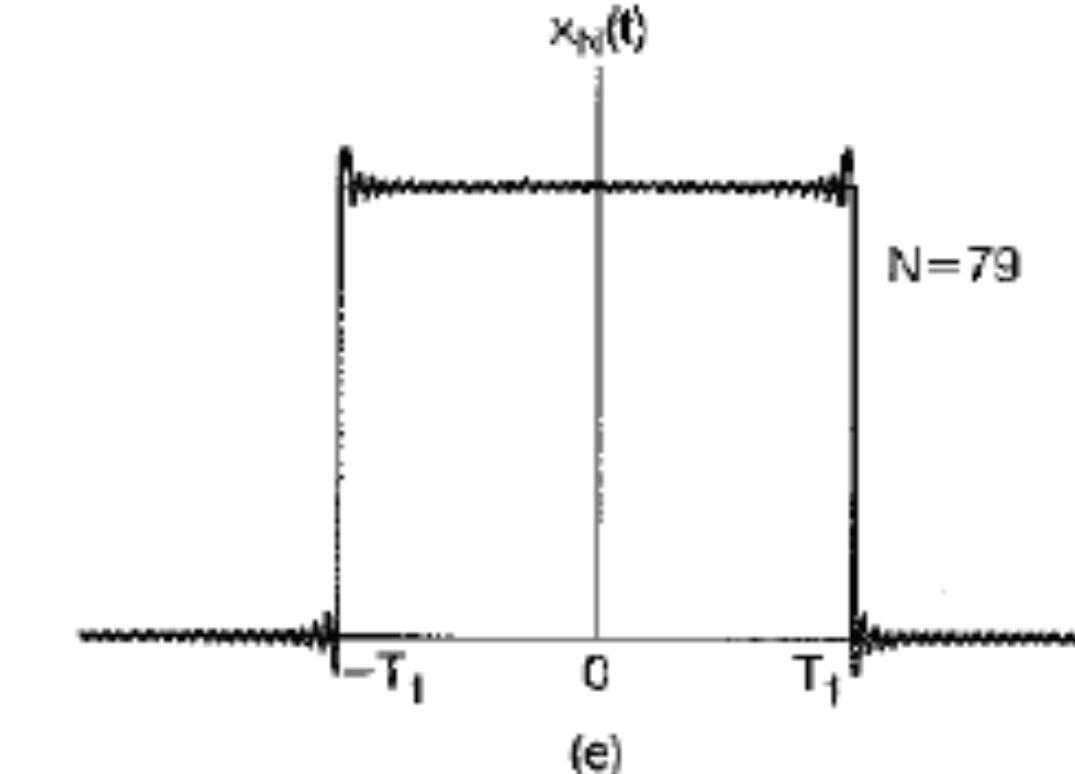
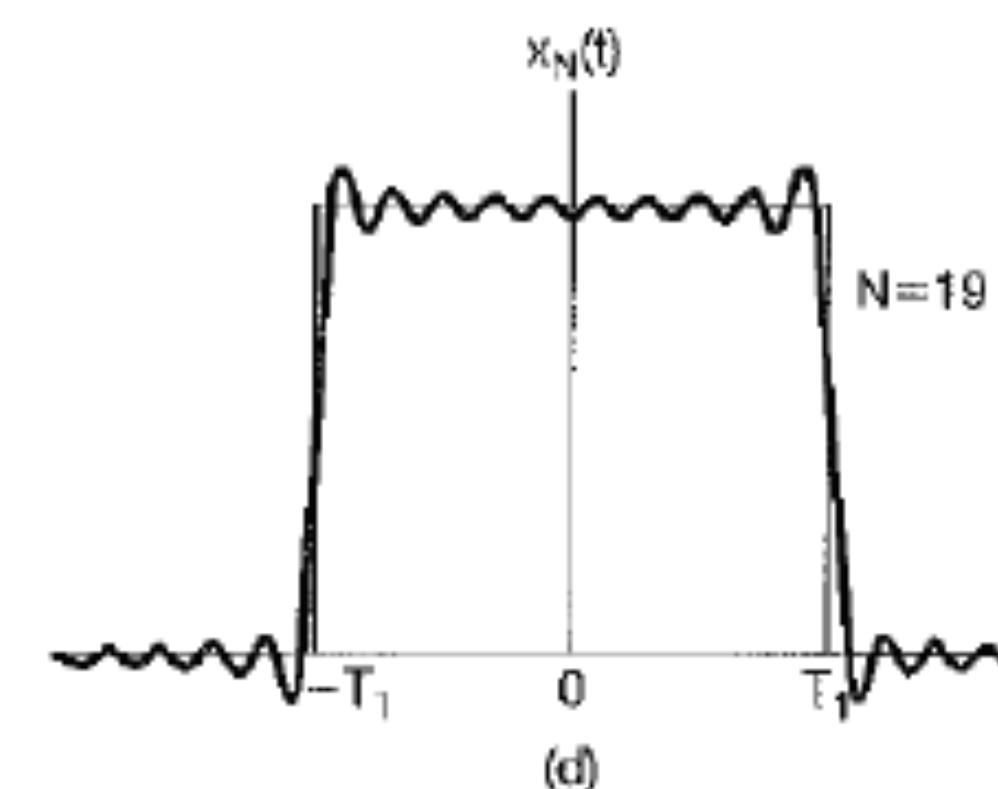
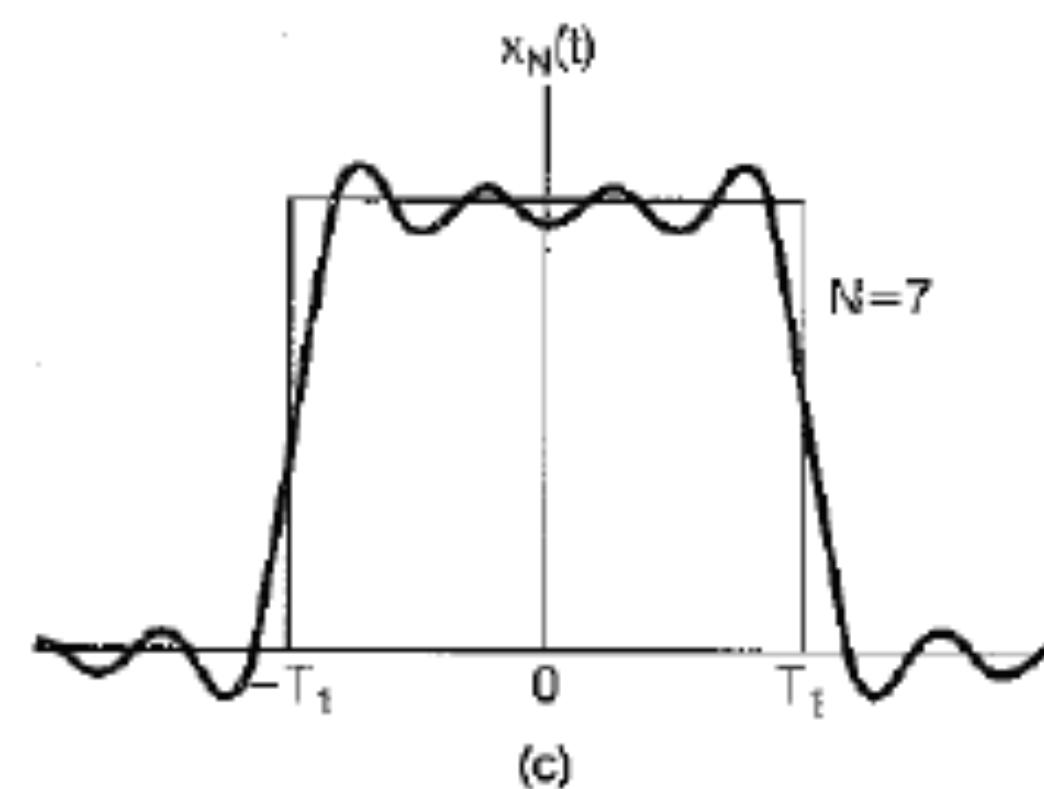
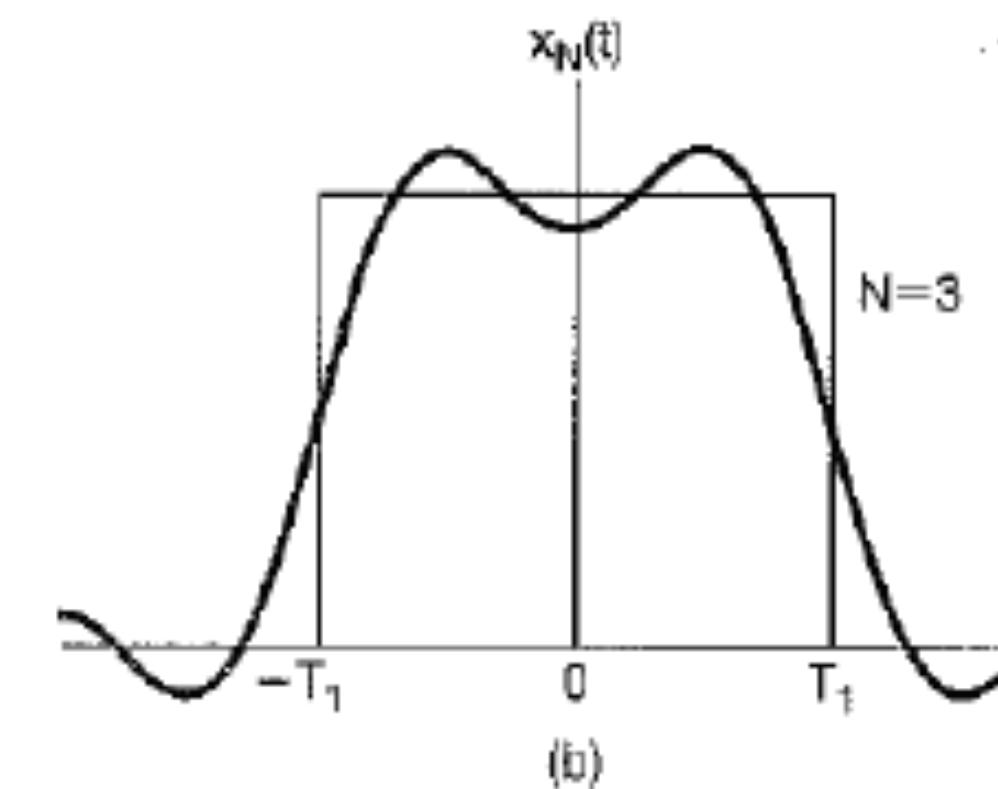
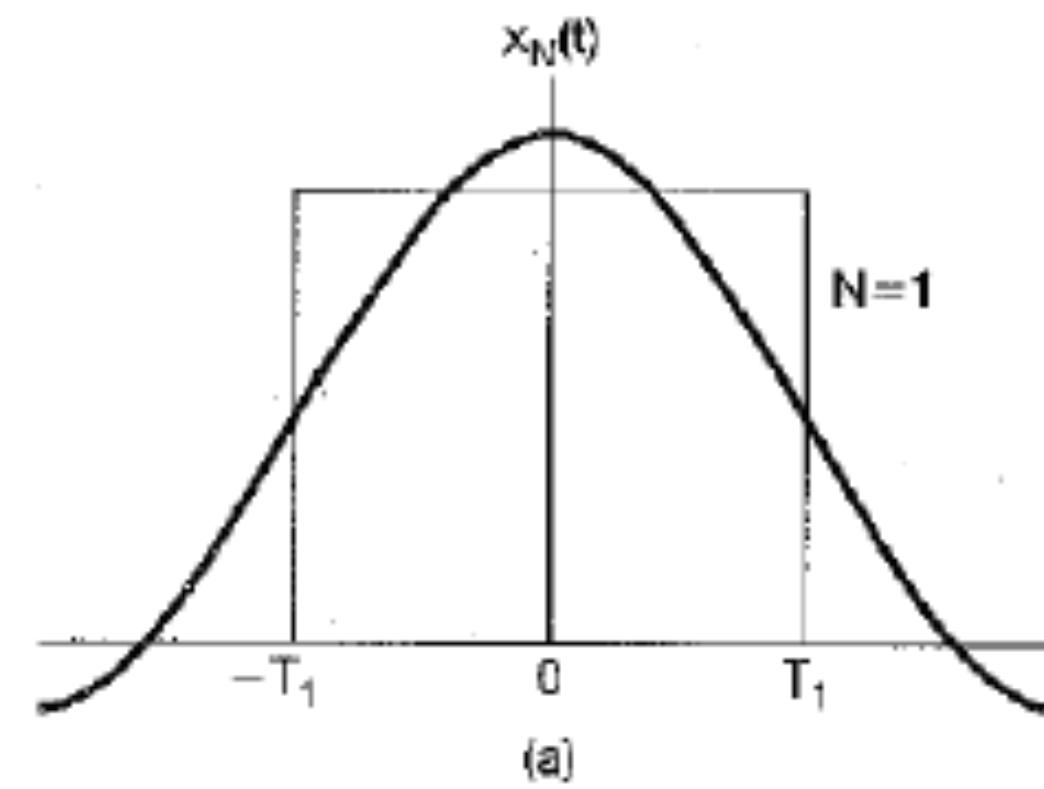
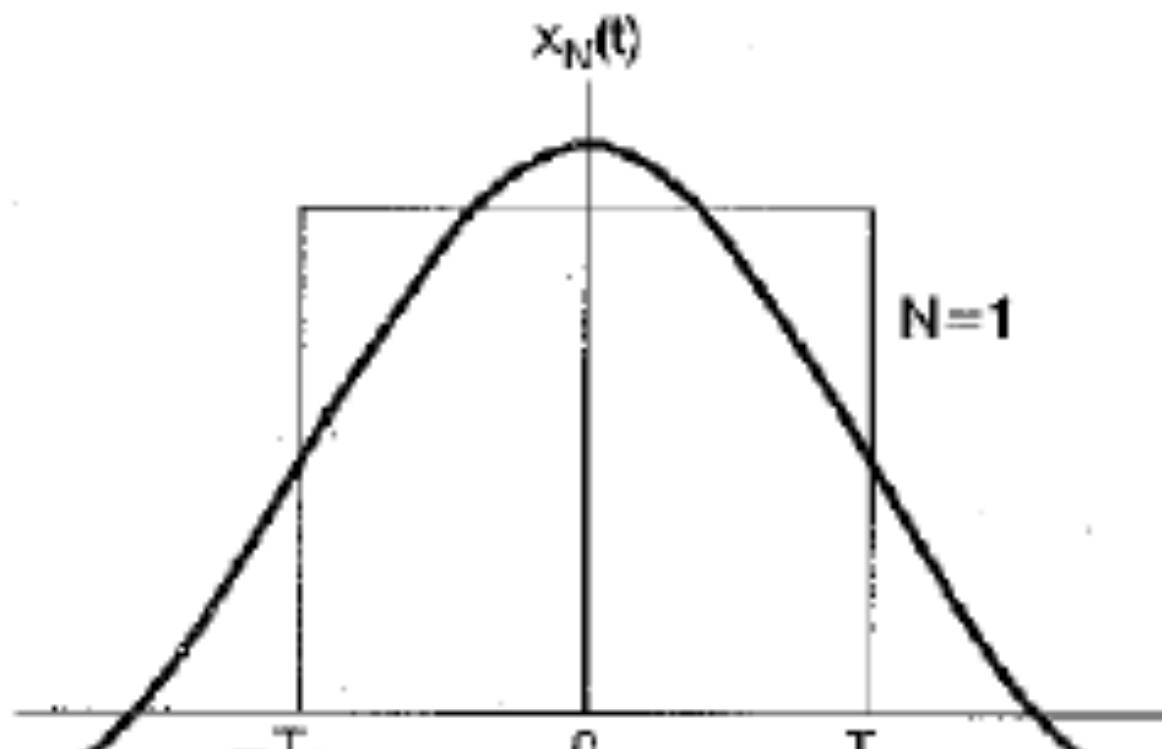
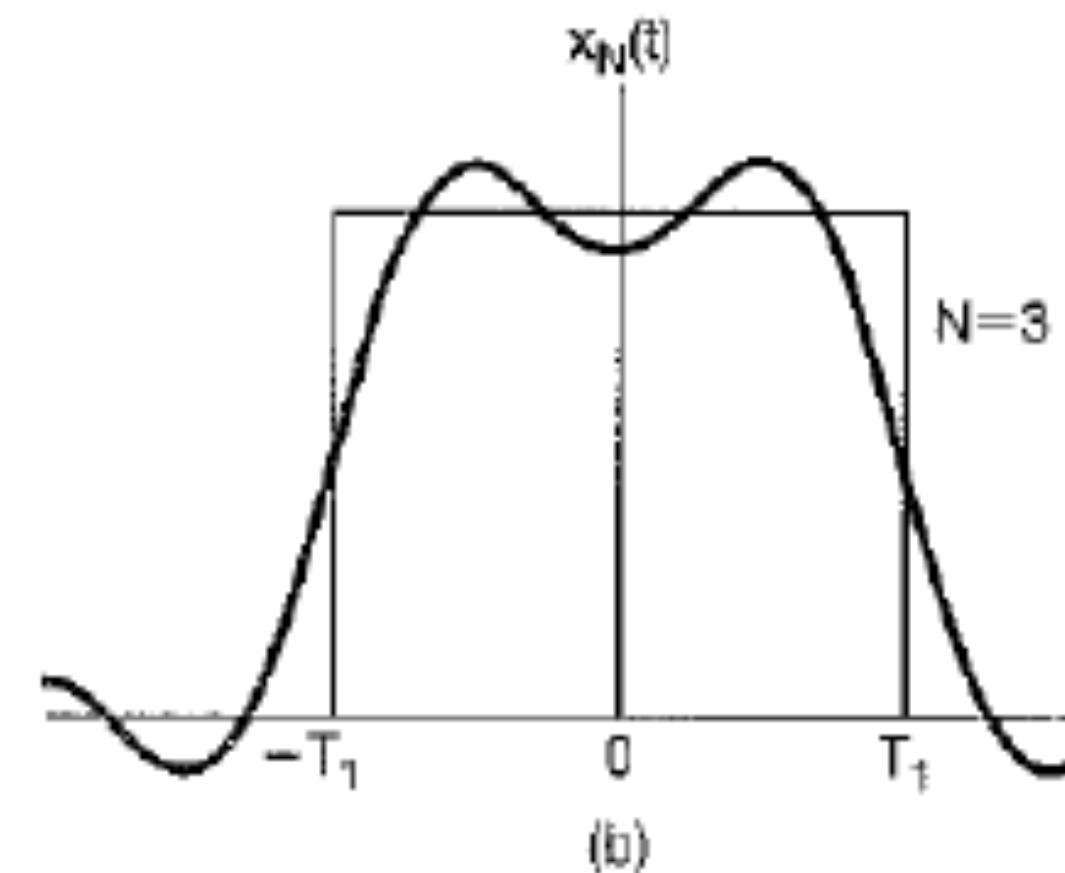


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^N a_k e^{ik\omega_0 t}$ for several values of N .

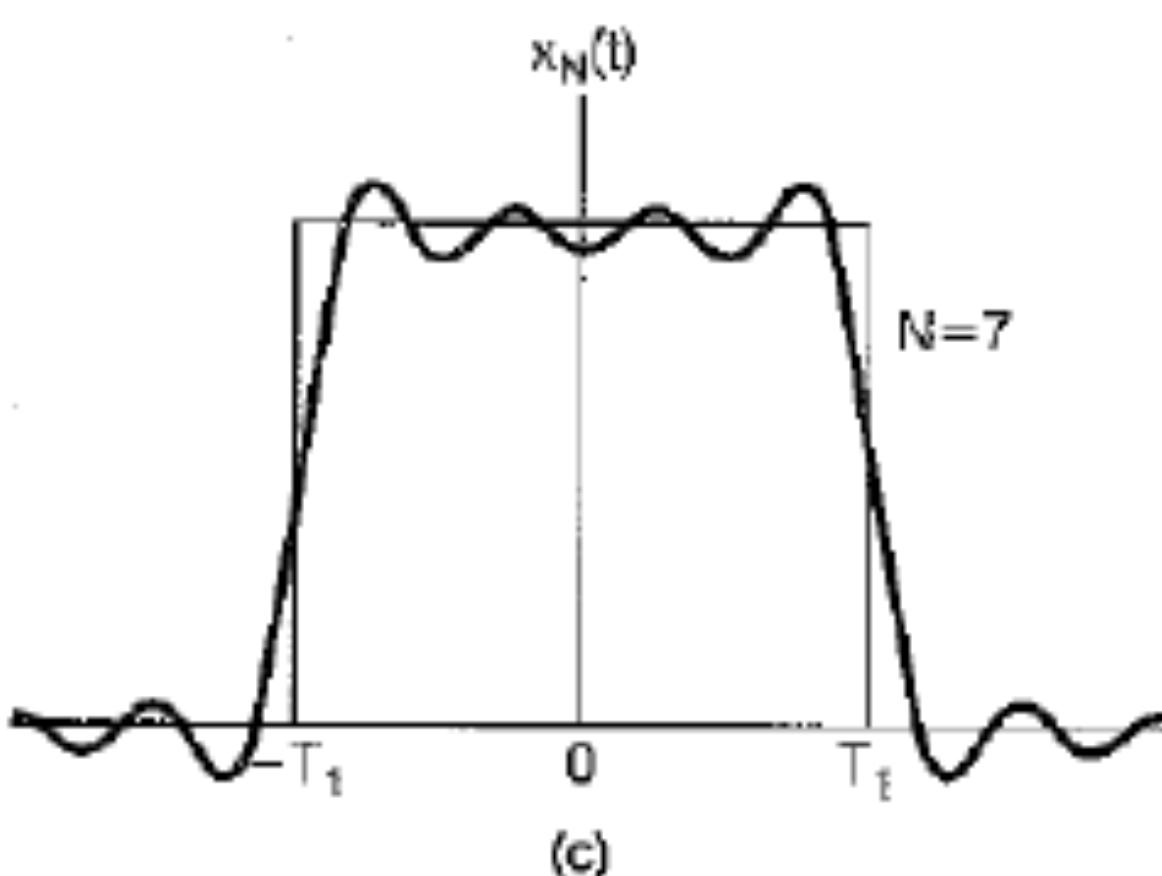
Convergence: Gibbs phenomenon



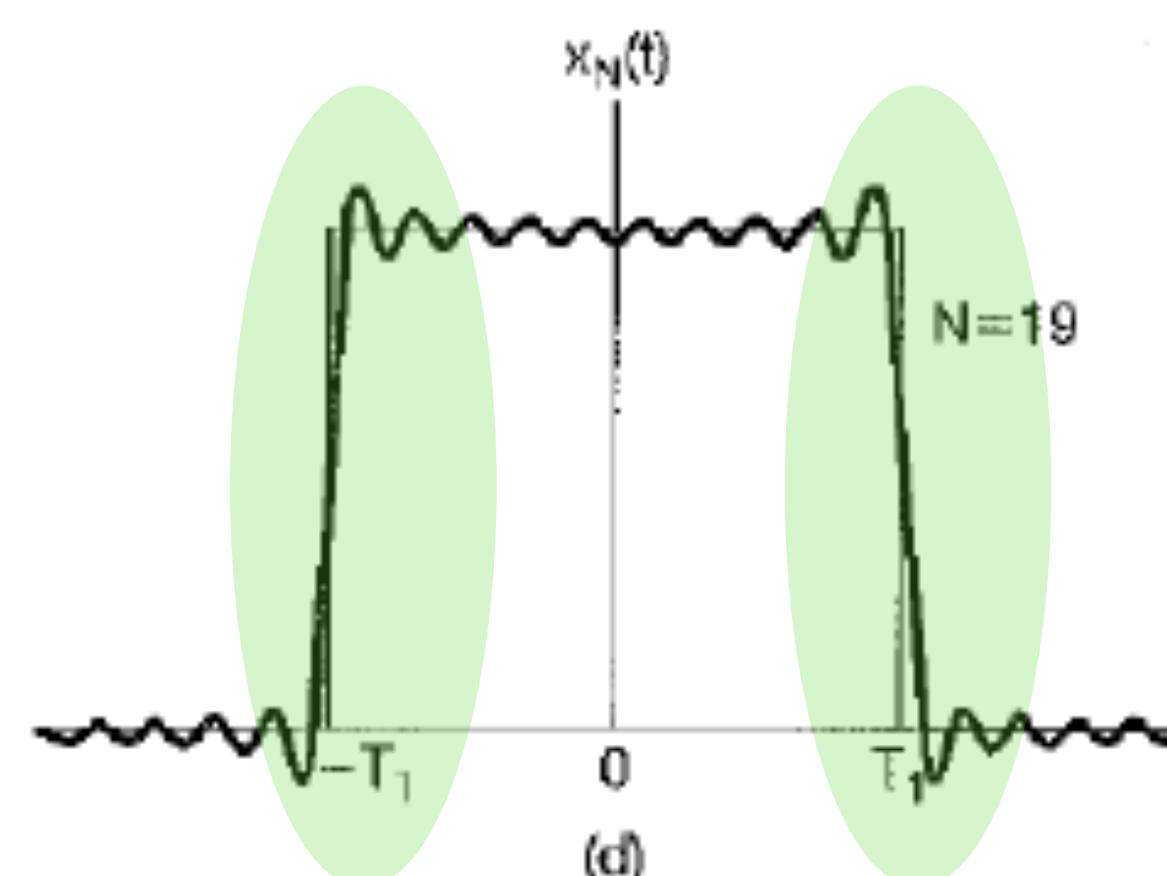
(a)



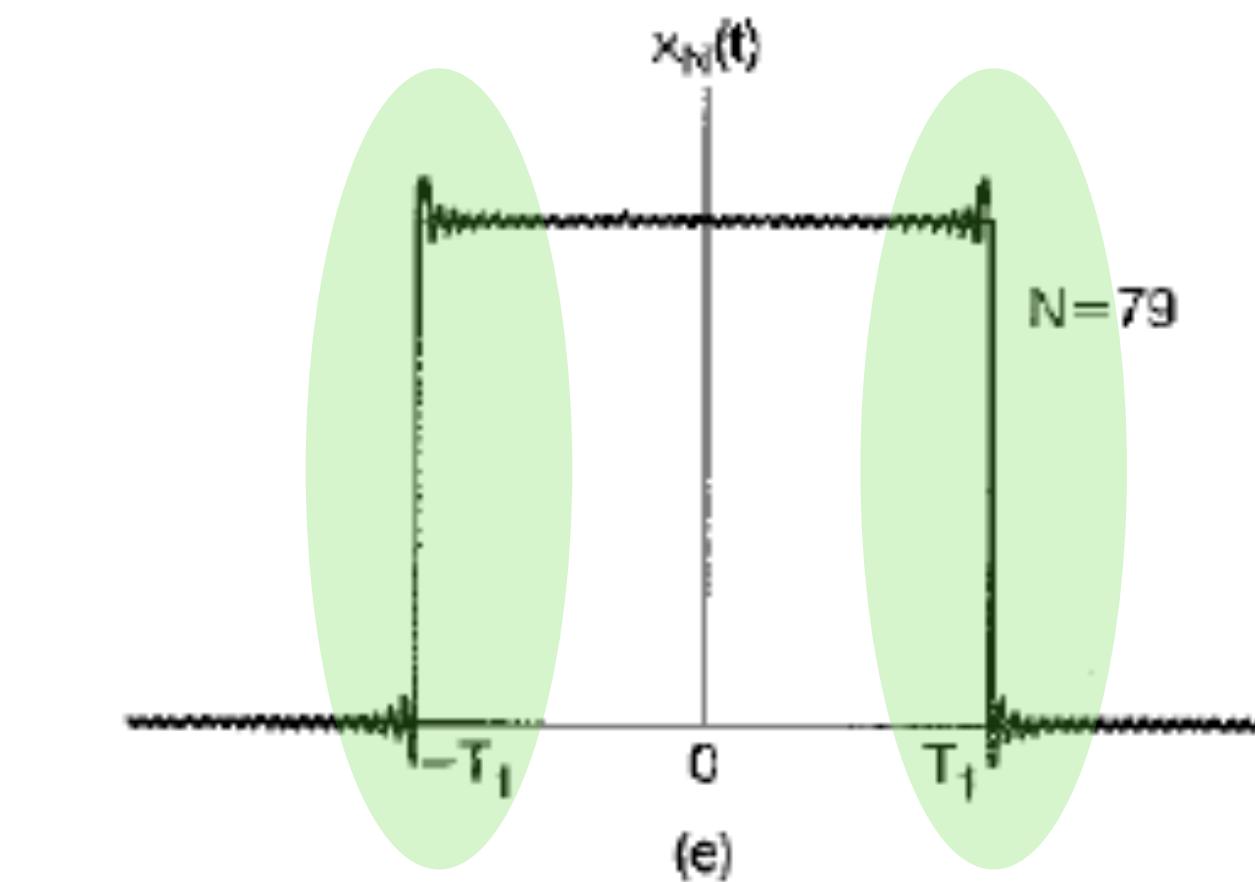
(b)



(c)



(d)



(e)

Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^N a_k e^{ik\omega_0 t}$ for several values of N .

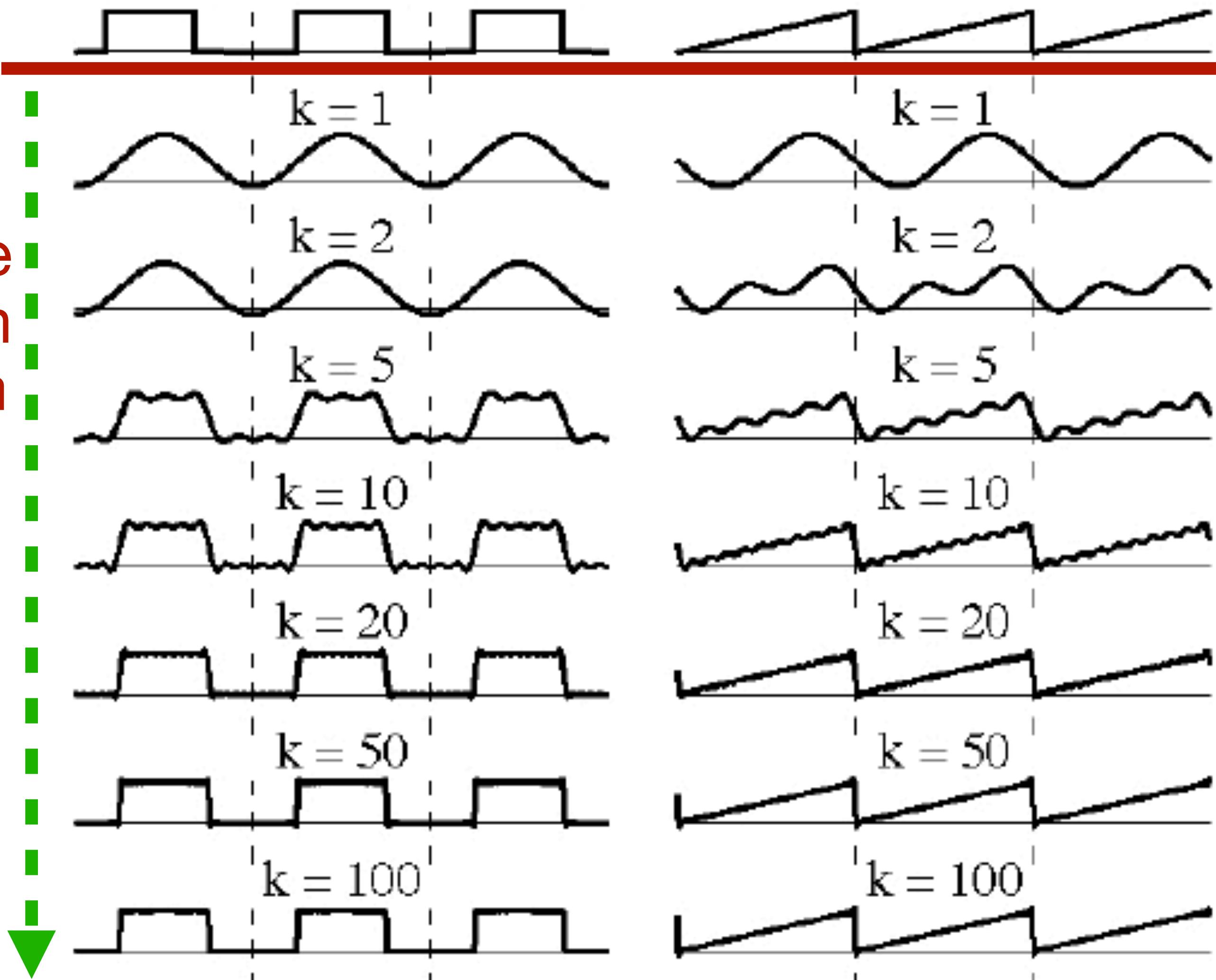
- some problem in the discontinuities

Convergence: examples

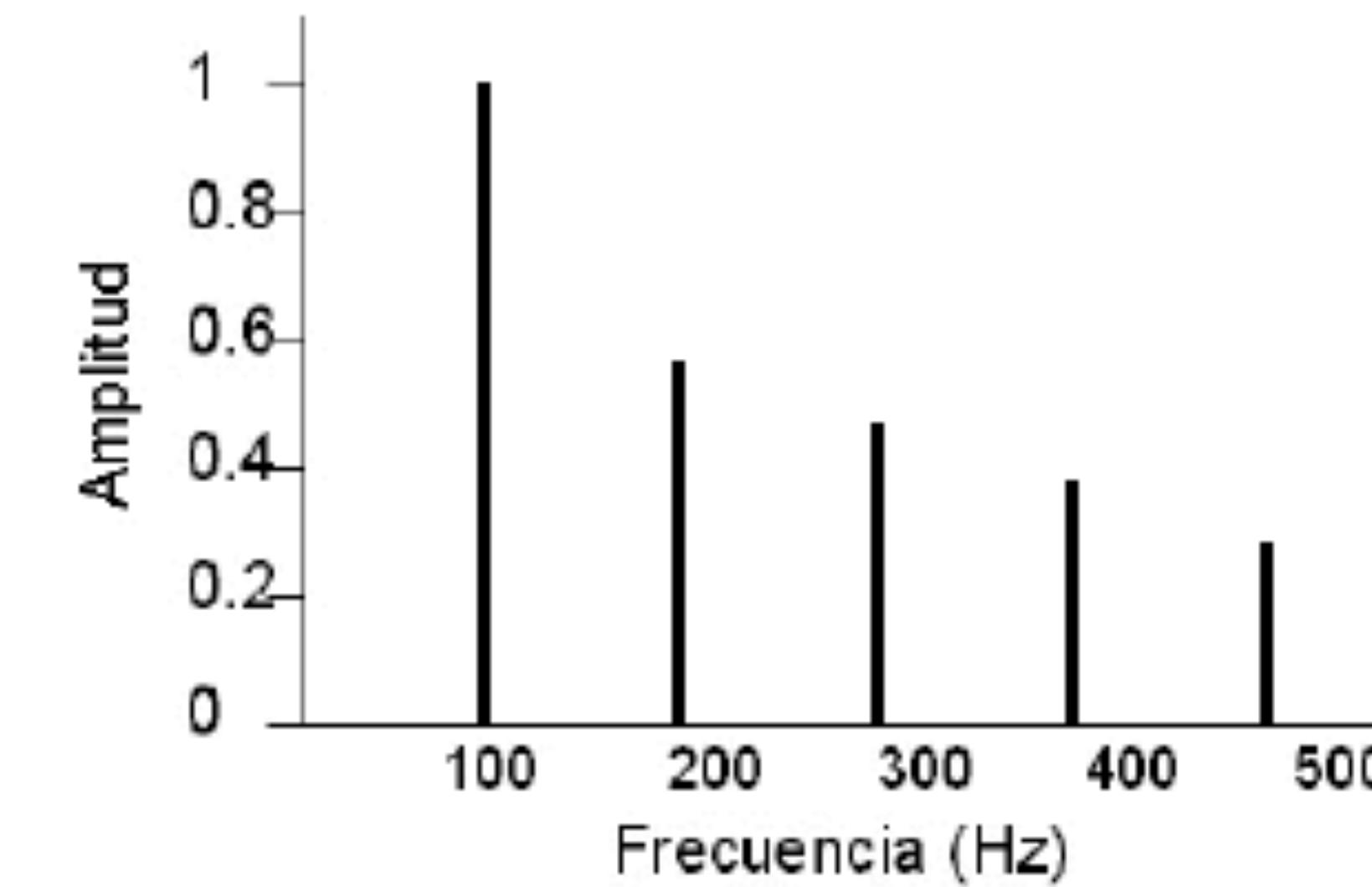
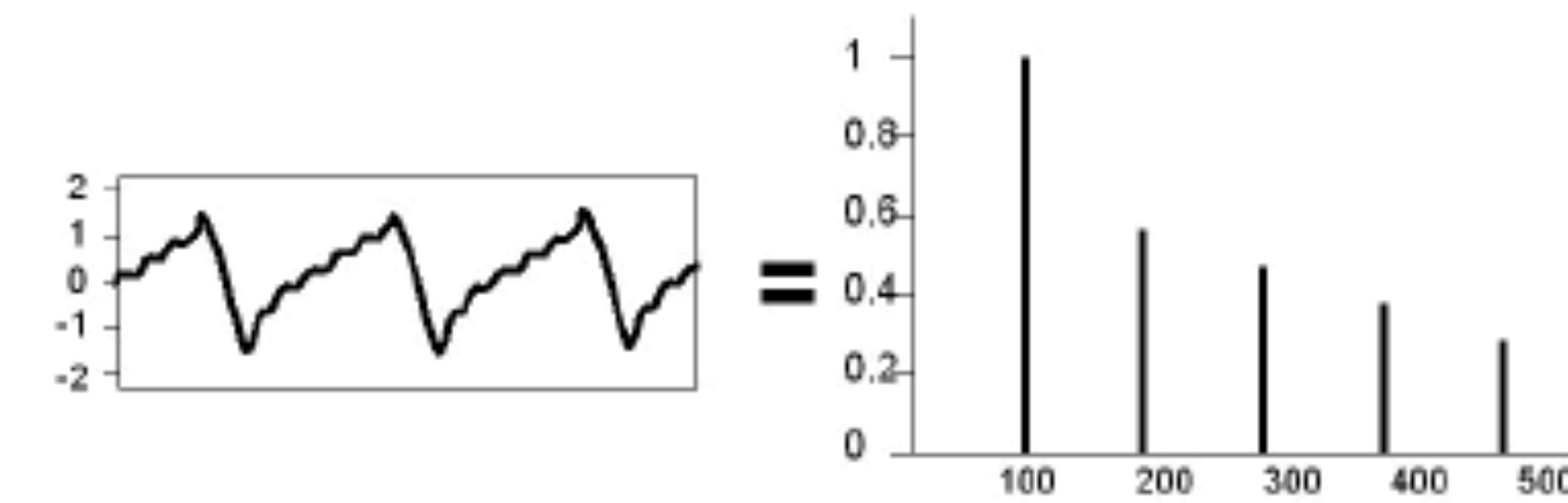
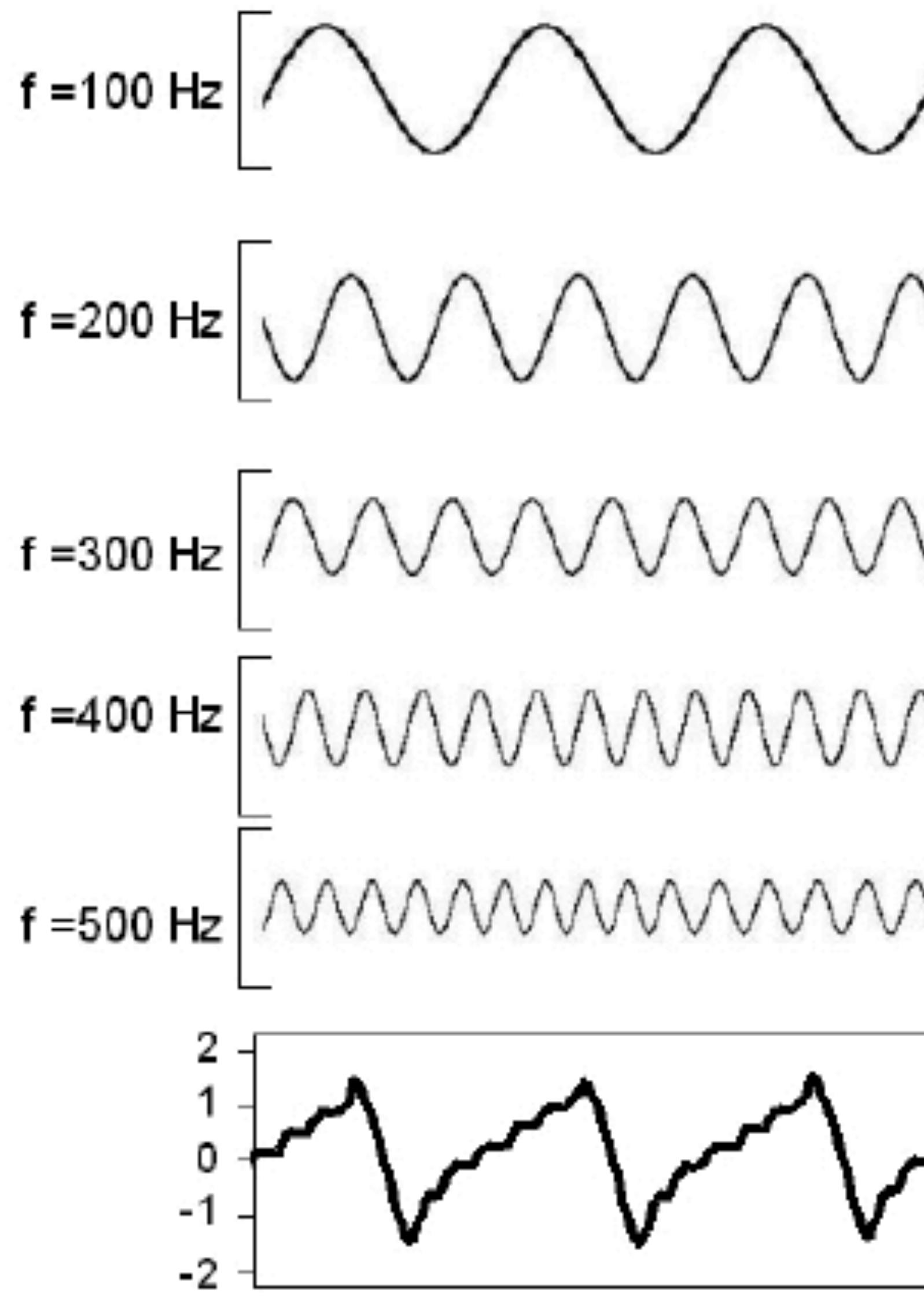
- Cualquier función periódica puede ser representada por la suma de senos y cosenos de diferentes amplitudes y frecuencias

- Signal:

Increasing the components in the truncated sum



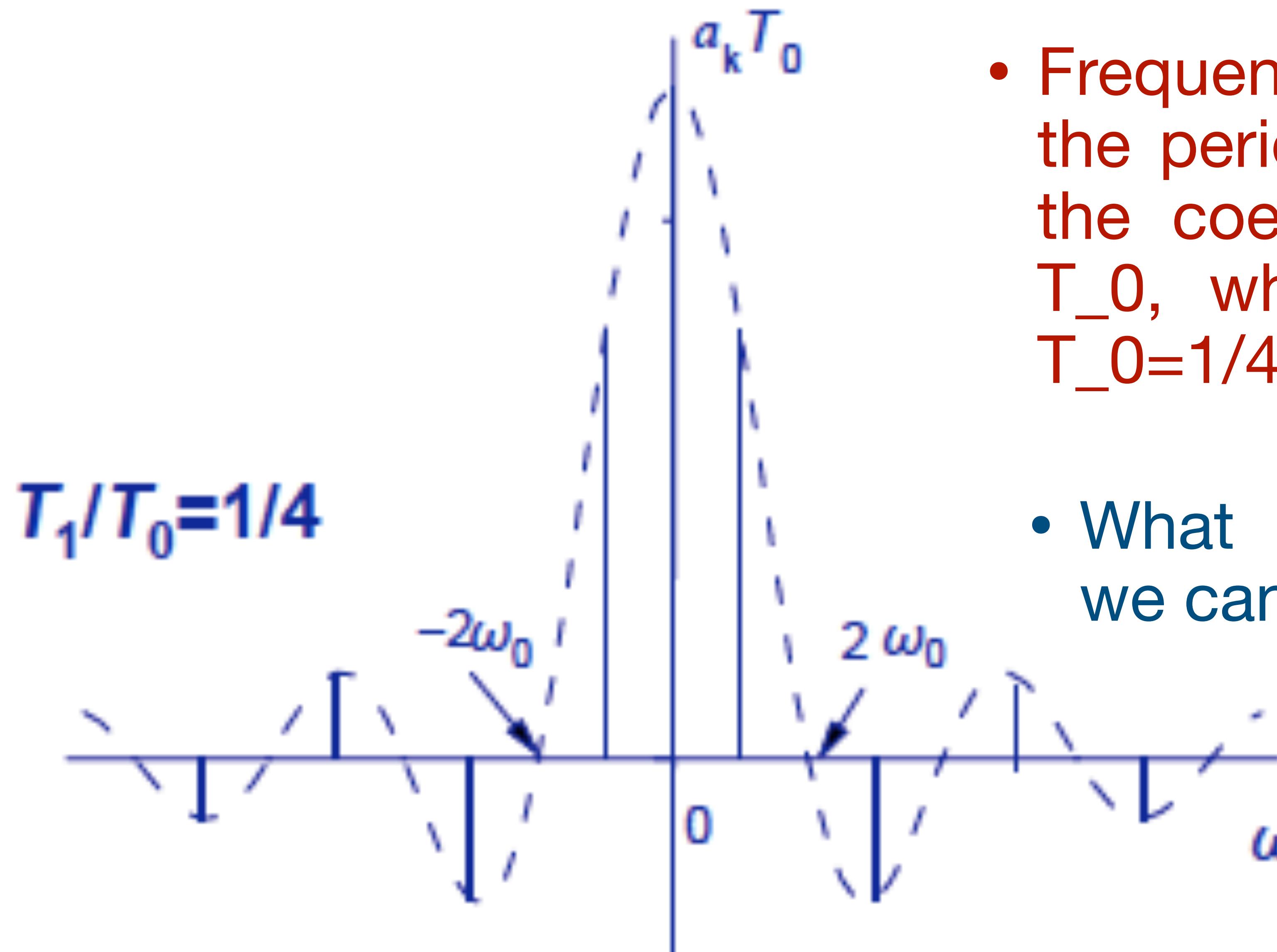
Convergence: examples



Again about the example of the rectangle

- We have saw that the a_k are real (this is due to $x(t)$ is real and even)
- Then, in this case we can represent directly a_k .
- For “normalization” reasons, we will plot $a_k T_0$ (a_k times T_0)

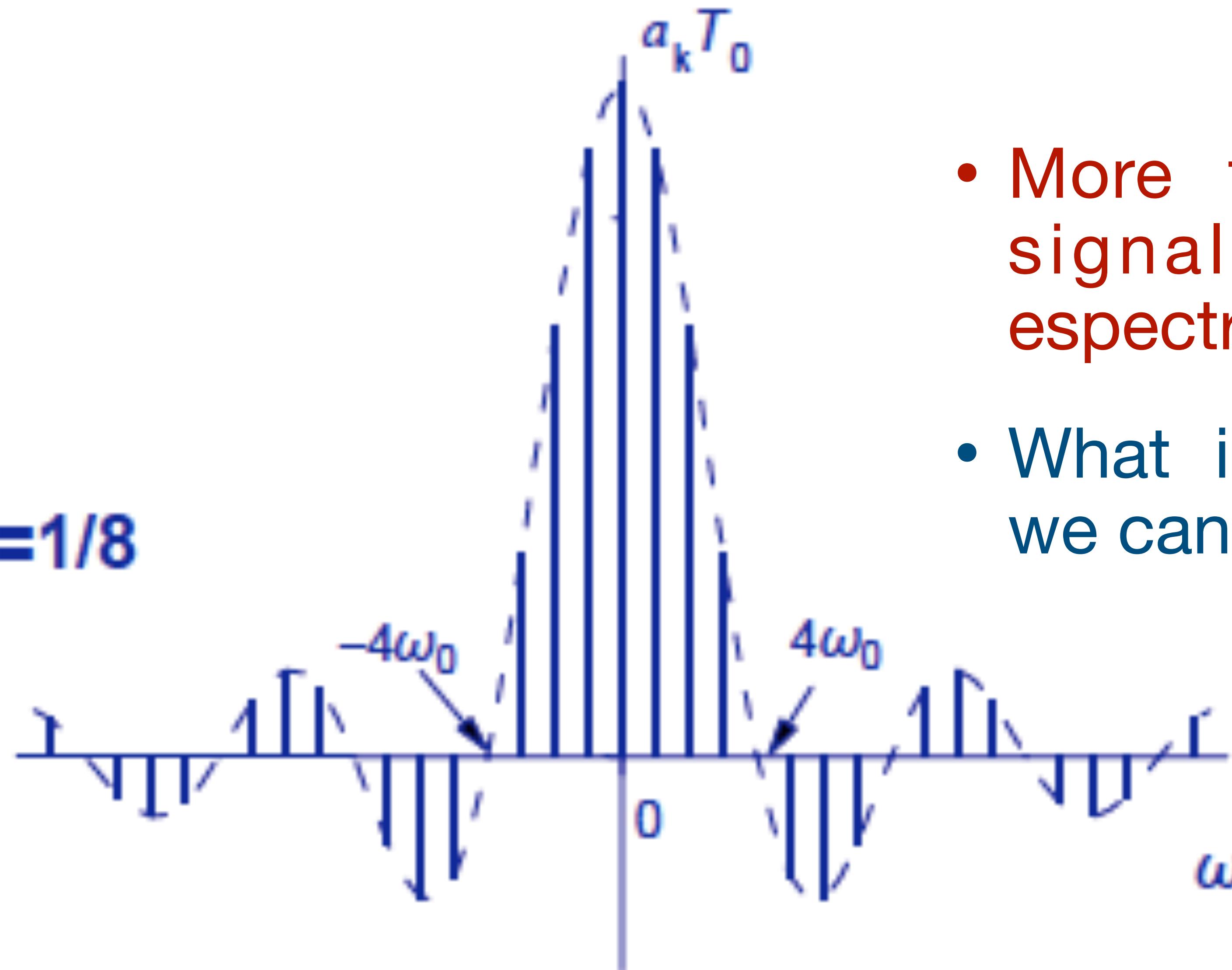
Again about the example of the rectangle



- Frequencies contained in the periodic signal $x(t)$ and the coefficients a_k times T_0 , when the ratio $T_1/T_0=1/4$
- What is the dashed line?
we cannot answer now...

And increasing the period T_0 ?

$$T_1/T_0 = 1/8$$

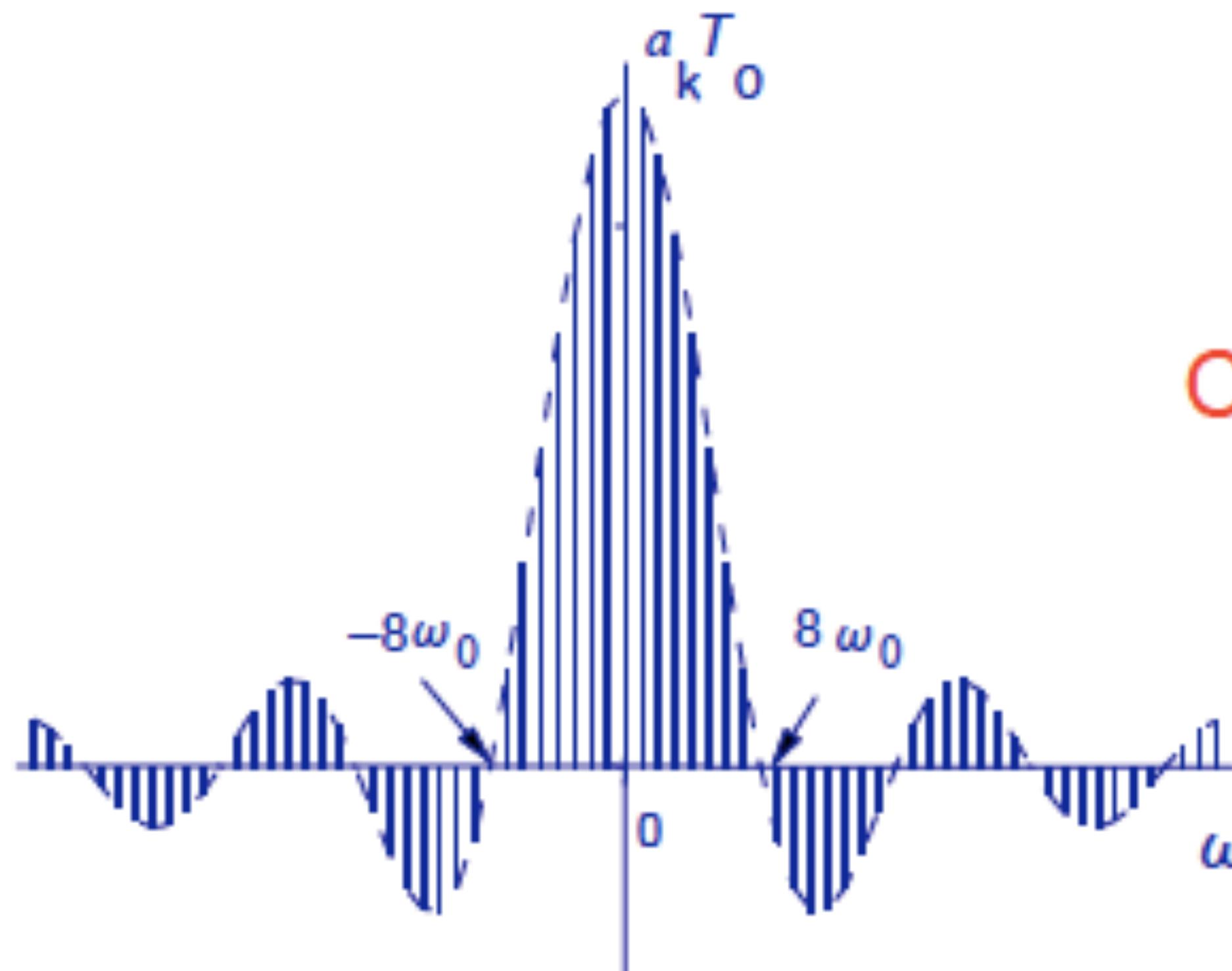


- More frequencies in the signal !!! “mas banda espectrales”
- What is the dashed line? we cannot answer now...

And again increasing the period T_0 ?

$$T_1/T_0 = 1/16$$

Que pasa cuando
 T_0 va a infinito?

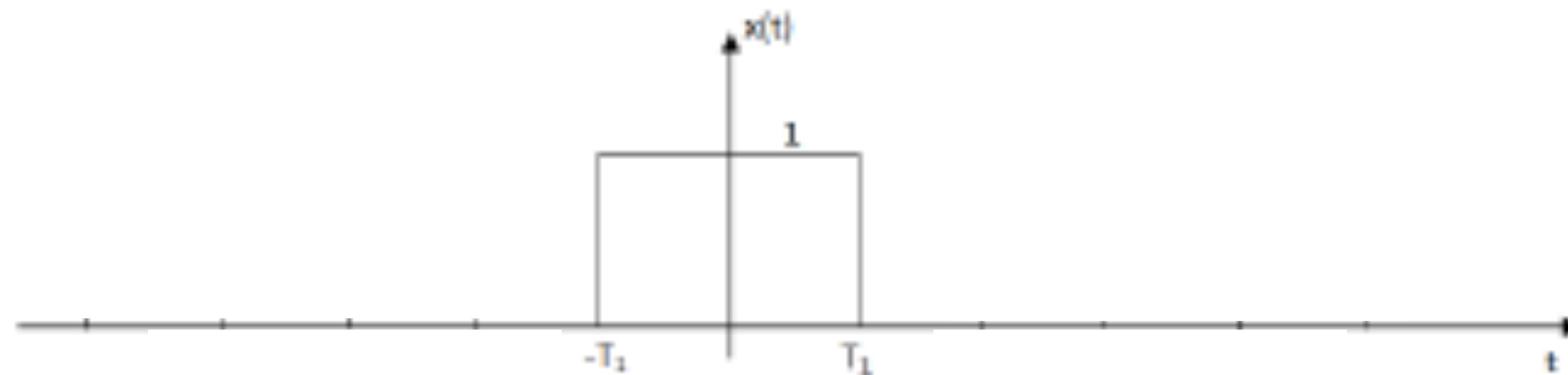


Conforme aumenta T_0 ,
aumenta el número
de componentes
espectrales

- More frequencies in the signal !!!
- What is the dashed line? we cannot answer now...

And again increasing the period T_0 ?

- what happens when T_0 diverges? i.e., T_0 goes to infinity?
- the signal becomes non-periodic....
- we will see it again, and we will see what is the dashed line...



Important property of a_k

- Supongamos que $x(t)$ es real $\rightarrow x(t)=x^*(t)$

$$a_k \implies \omega_k = k\omega_0$$

$$x(t) = \sum_k a_k e^{j\omega_k t} = \left(\sum_k a_k e^{j\omega_k t} \right)^* = x^*(t) = \sum_k a_k^* e^{-j\omega_k t} = \sum_k a_{-k}^* e^{j\omega_k t}$$

- Luego, si la señal es real, los coeficientes de la serie de Fourier verifican:

$$a_k = a_{-k}^*$$

- Los coeficientes poseen **antisimetría conjugada**, o lo que es lo mismo, son **hermíticos**

Important property of a_k

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longrightarrow x^*(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

$k = -k'$

$$x^*(t) = \sum_{k'=-\infty}^{+\infty} a_{-k'}^* e^{jk'\omega_0 t}$$

Important property of a_k

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad x^*(t) = \sum_{k'=-\infty}^{+\infty} a_{-k'}^* e^{jk'\omega_0 t}$$

**...but k' or k are just labels (etiquetas) !!!
then we can write:**

$$x^*(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

Important property of a_k

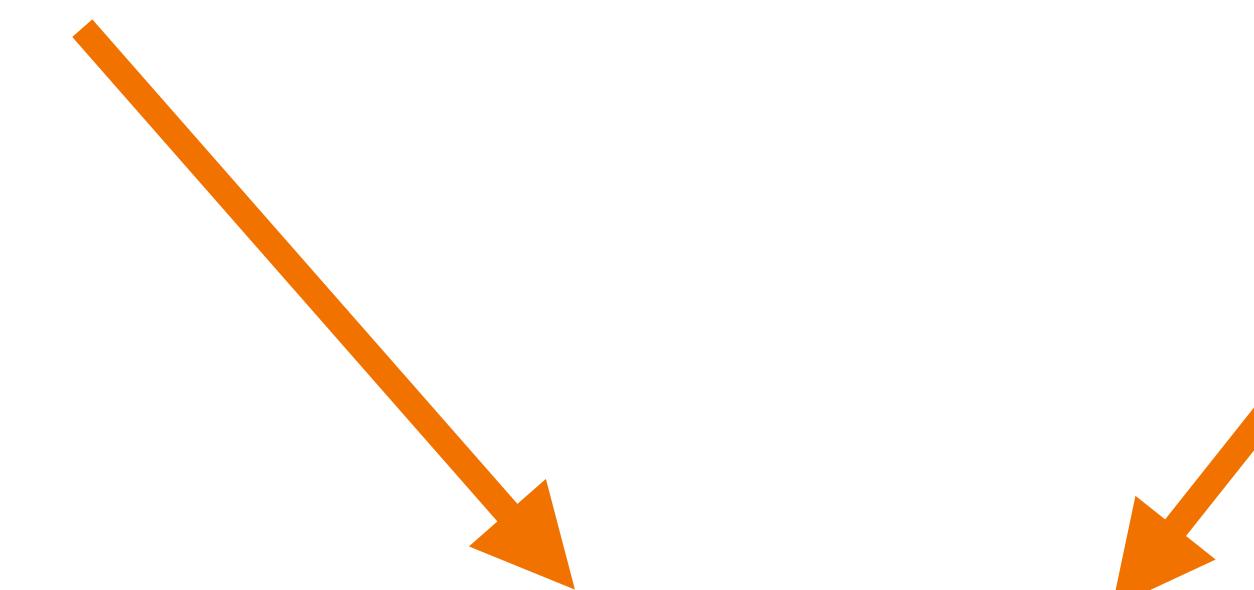
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

if $x(t) = x^*(t)$:

$$a_k = a_{-k}^*$$

$$a_k^* = a_{-k}$$



Important property of a_k

- Then if $x(t)$ is a real signal, we have:

$$a_k = a_{-k}^*$$

$$a_k^* = a_{-k}$$

Other alternative form of the Fourier Series for a periodic **real** signal

- ALWAYS we can write:

$$x(t) = a_0 + \sum_{k=1}^{+\infty} (a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t})$$

- If $x(t)$ is a real signal, we have:

$$x(t) = a_0 + \sum_{k=1}^{+\infty} (a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t})$$

Other alternative form of the Fourier Series for a periodic **real** signal

- If $x(t)$ is a real signal, we have:

$$x(t) = a_0 + \sum_{k=1}^{+\infty} (a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t})$$

$$x(t) = a_0 + 2 \sum_{k=1}^{+\infty} \text{Real}\{a_k e^{jk\omega_0 t}\}$$

Other alternative form of the Fourier Series for a periodic **real** signal

- If $x(t)$ is a real signal, and expressing the coefficients in *polar form*,

$$a_k = A_k e^{j\phi_k}$$

$$x(t) = a_0 + 2 \sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \phi_k)$$

Other alternative form of the Fourier Series for a periodic **real** signal

- If $x(t)$ is a real signal, and expressing the coefficients in *rectangular form*,

$$a_k = B_k + jC_k$$

$$x(t) = a_0 + 2 \sum_{k=1}^{+\infty} \text{Real} \{ B_k \cos(k\omega_0 t) + jB_k \sin(k\omega_0 t) + jC_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t) \}$$

$$x(t) = a_0 + 2 \sum_{k=1}^{+\infty} B_k \cos(k\omega_0 t) - 2 \sum_{k=1}^{+\infty} C_k \sin(k\omega_0 t)$$

Example: Fourier series of the cosine

$$x(t) = \cos(\omega_0 t)$$



- For Euler...

$$\cos(\omega_0 t) = \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}]$$



- Sum of complex exponentials...
this is already the Fourier Series !!!

Example: Fourier series of the cosine

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$$a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2},$$

$$a_k = 0 \text{ for all } k \neq -1, 1$$

Example: Fourier series of the cosine

- Since the cosine is a real signal, check that:

$$a_k = a_{-k}^*$$

Example: Fourier series of the sine

$$x(t) = \sin(\omega_0 t)$$



- For Euler...

$$\sin(\omega_0 t) = \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}]$$



- Sum of complex exponentials...
this is already the Fourier Series !!!

Example: Fourier series of the cosine

$$\sin(\omega_0 t) = \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}]$$



- Sum of complex exponentials...
this is already the Fourier Series !!!

$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0 \text{ for all } k \neq -1, 1$$

Example: Fourier series of the cosine

- Since the sine is a real signal, check that:

$$a_k = a_{-k}^*$$

Does The Fourier series exist always?

- The Fourier series decomposition exists *almost for all* the periodic signals.
- The integral of synthesis equation must exist:

$$a_{\color{red}k} = \frac{1}{T_0} \int_{T_0} x(t) e^{-j \color{red}k \omega_0 t} dt$$

since it is computed in a finite region - such as $[0, T_0]$ - there are “a lot of chances” that this integral exists and is a finite value, for all k .

Does The Fourier series exist always?

- The integral of synthesis equation must exist:

$$a_{\color{red}k} = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\color{red}k\omega_0 t} dt$$

However, for some ver strange signal $x(t)$, this integral does not exists.

Conditions for the existence of the Fourier series

- A class of periodic signals that admits Fourier Series, is the family of all the periodic signals with finite energy (in the period) - below we denote the signal as $f(t)$:

$$\int_{T_0} |f(t)|^2 dt < +\infty$$

- convergence in all “places” (for almost all t) maybe expect some points “ t ” (isolated points - of null energy/measure - “aislados”).

Conditions for the existence of the Fourier series

ALTERNATIVELY:

- Considering a signal $f(t)$, the Dirichlet conditions are:

1. f must be **absolutely integrable** over a period.
2. f must be of **bounded variation** in any given bounded interval.
3. f must have a finite number of **discontinuities** in any given bounded interval, and the discontinuities cannot be infinite.

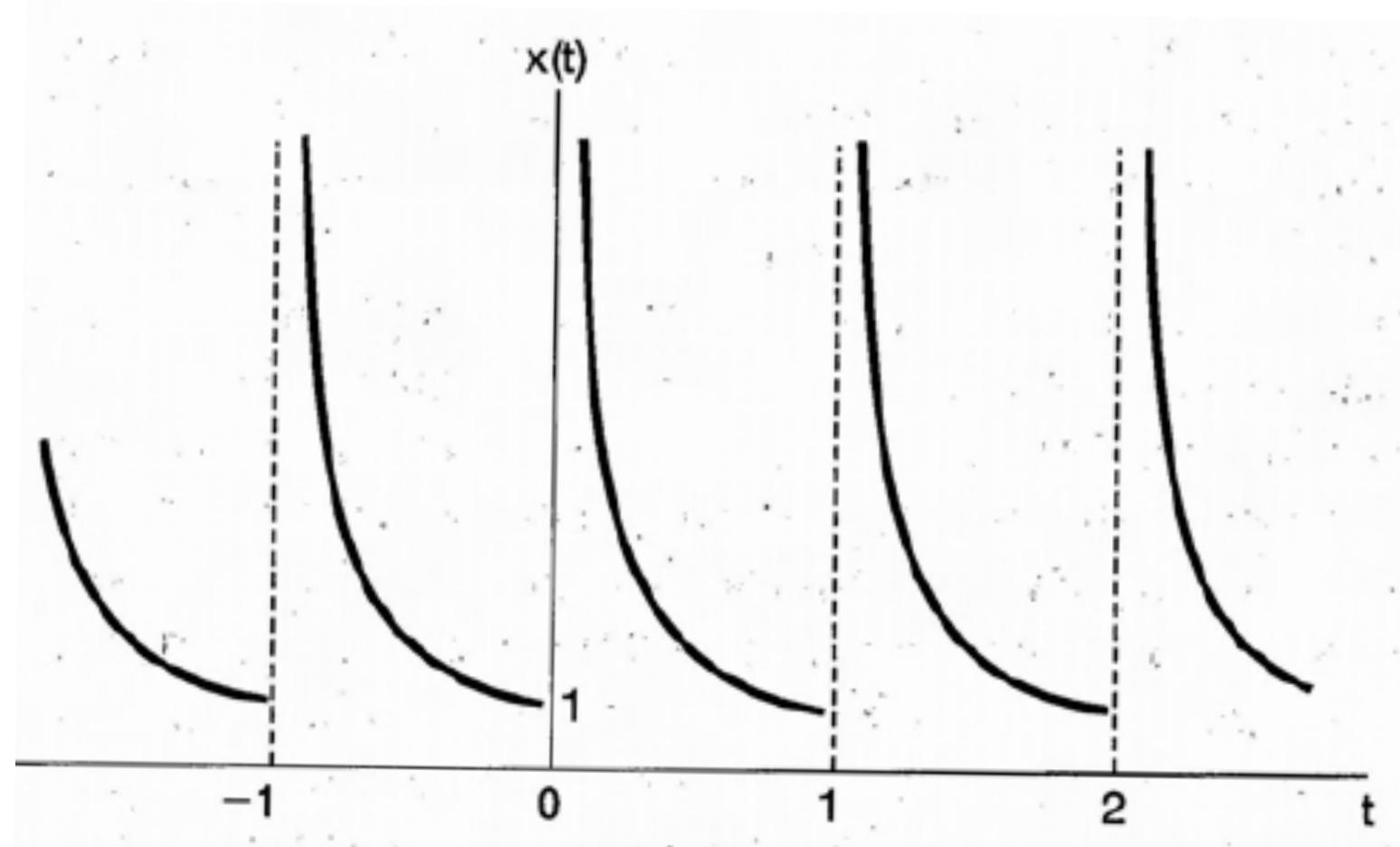

$$\int_{T_0} |f(t)|dt < +\infty$$

Conditions for the existence of the Fourier series

- **Example 1:** Consider the signal with period $T_0=1$:

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1;$$

- It does not satisfy the first Dirichelet condition (viola la primera condición)



Conditions for the existence of the Fourier series

- **Example 2:** an example of signal that fulfills the first condition but does not fulfill the second condition is

Un ejemplo de una función de tiempo que cumple con la condición 1 pero no la condición 2 es

$$x(t) = \operatorname{sen}\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1,$$

Para esta función, la cual es periódica con $T = 1$,

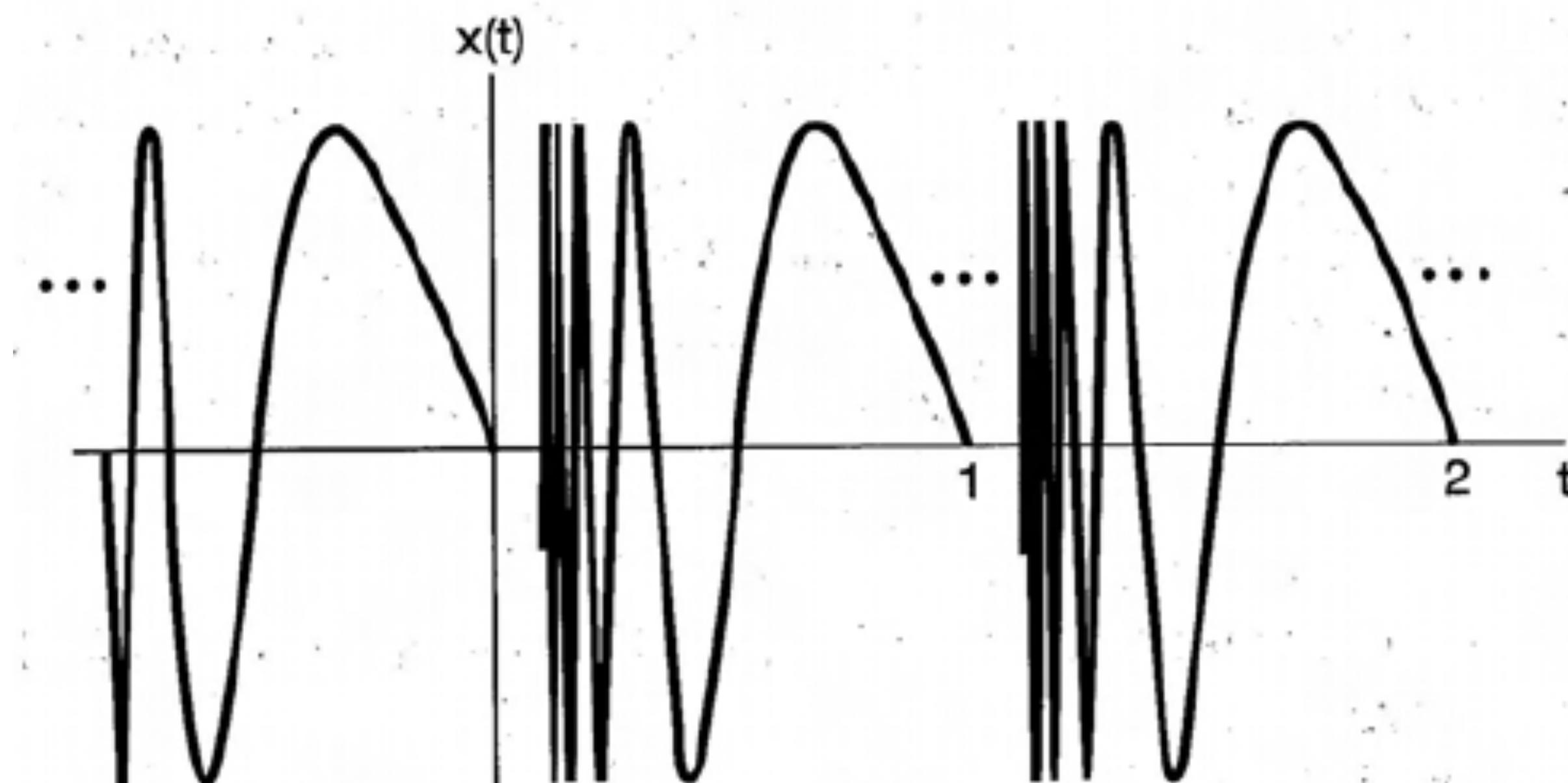
$$\int_0^1 |x(t)| dt < 1.$$

Sin embargo, la función tiene un número infinito de máximos y mínimos en el intervalo.

It is amazing that the integral in the period is finite...think on it !!

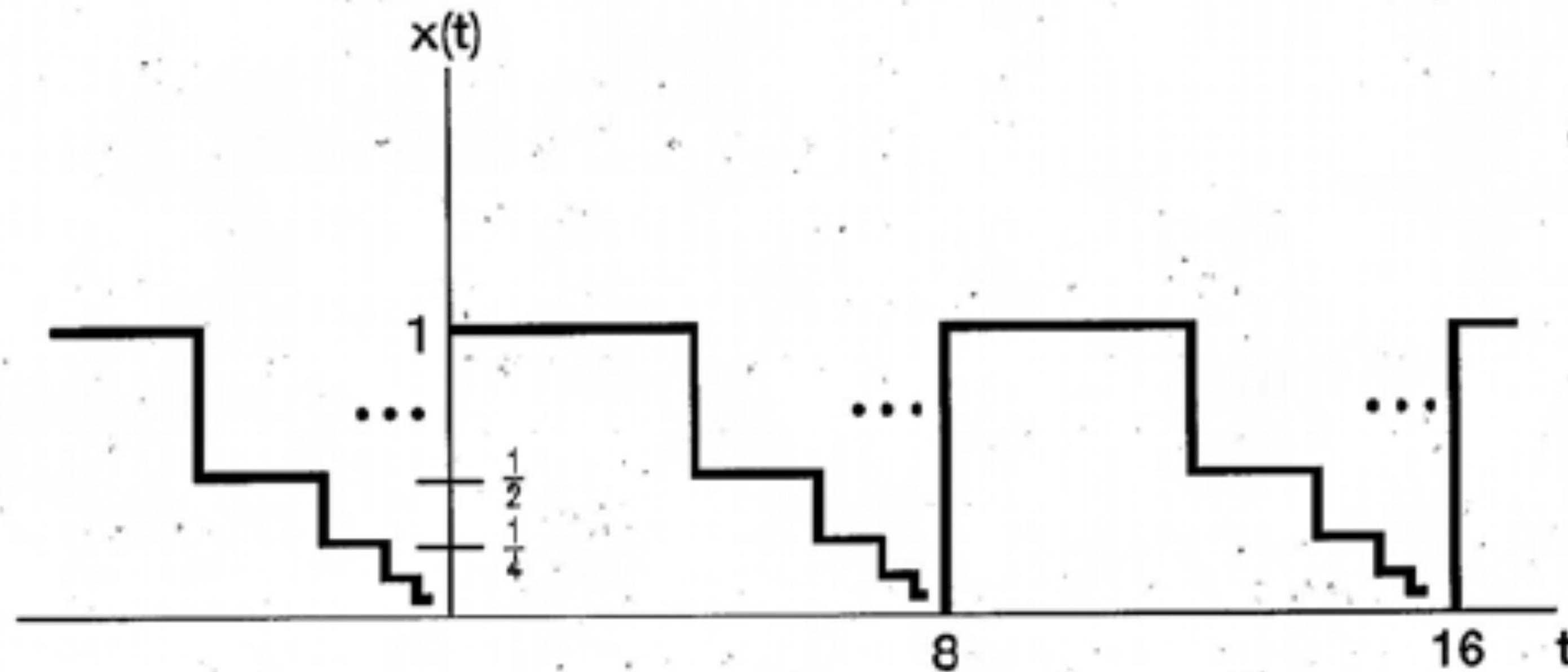
Conditions for the existence of the Fourier series

- **Example 2:**



Conditions for the existence of the Fourier series

- **Example 3:** an example of signal that does not fulfill the third condition is



Questions?